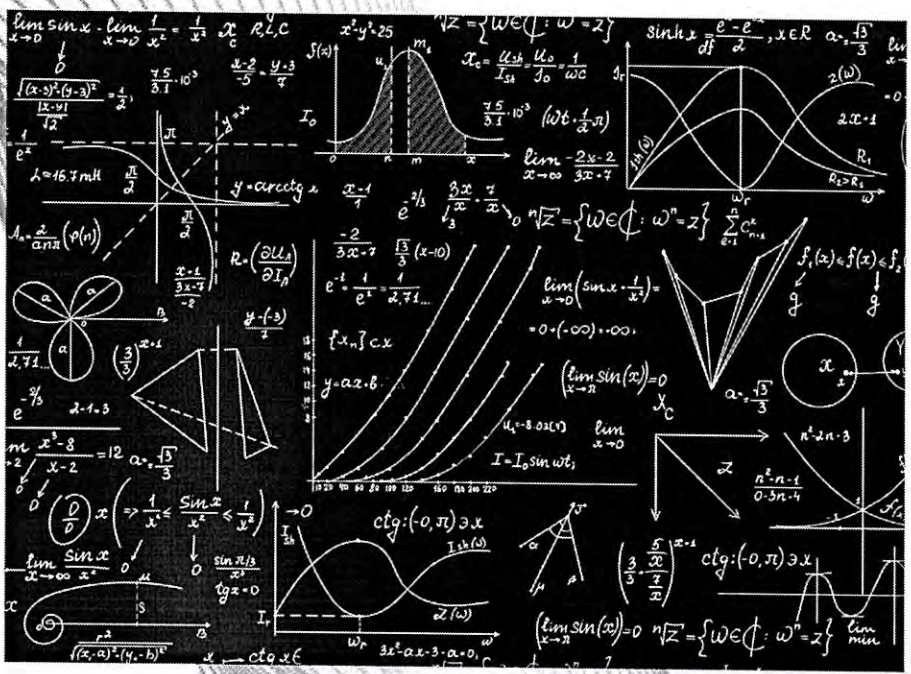
ADVANCES IN MATHEMATICAL ANALYSIS AND ITS APPLICATIONS



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Advances in Mathematical Analysis and its Applications

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Chapter 10

Applications of differential transform method on some functional differential equations

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10.1 Introduction

The functional differential equations with proportional delay were first studied by Ockendon and Taylor [18] in their work of collecting electric current for the pantograph of an electric locomotive, hence named pantograph equations. The investigation of these equations is important since they find applications in economic activities involving production, planning and decision

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making, number theory, biological phenomena, probability concepts applied on algebraic structures, electrodynamics, quantum mechanics, nautical science, and astrophysics, among others [2, 6, 14, 15]. Further, time delays are significant in engineering problems such as feedback loops equipped with sensors and actuators, the transmission of signals to remote center, in predictions and control systems, etc. [1, 9, 11, 16, 28].

Several analytical and numerical methods have been proposed by many researchers to study solutions of proportional delay differential equations (PDDEs) which include Adomian decomposition method (ADM) [5], Homotopy perturbation method (HPM) [26], Homotopy analysis method (HAM) [12], Variational iteration method (VIM) [8, 29], Taylor series [25], Runga-kutta method [30] and Collocation method [4]. Due to calculation of Adomian polynomials in ADM, evaluation of integrals in HPM, finding Lagrangian multipliers in VIM and discretization of variables and complex calculations in numerical methods make them unsuitable. We propose a simple approach involving the differential transformation in this chapter. The differential transformation has been introduced by G. Pukhov as the "Taylor transform" in 1976 and applied to the study of electrical circuits [19]. The differential transformation is closely related to Taylor expansion of real analytic functions. It has applications in solving different types of problems for all classes of differential equations (ordinary, partial, delayed, fractional, fuzzy etc.). The recent development and applications of DT are discussed in [3, 7, 13, 17, 20, 21, 23, 24] and references therein.

In the present chapter, the differential transformation is applied to solve proportional delay differential equations. The nonlinearity in the problems is addressed by using the partial ordinary Bell polynomials in the Faà di Bruno's formula. The results obtained by this technique are compared with analytical solutions. Detailed error analysis is provided. However, to the best of our knowledge, no researcher has applied the DTM using Bell polynomials on the practical problems discussed in the Section 10.3.

The chapter is organized as follows. In Section 10.2, we introduce the main idea and basic formulae of the differential transformation and provide necessary results to handle nonlinearity using partial ordinary Bell polynomials and discuss convergence results. Applications of the method are presented in Section 10.3.

10.2 Preliminaries

In this section, we discuss the main idea and basic formulae of the differential transformation as well as notation and results related to transformation of general nonlinear terms.

10.2.1 Definition of differential transform

Let $w(v)$ be a real analytical function in a domain Ω and $v = v_0$ be an arbitrary point in Ω. Then, $w(v)$ can be expanded in a Taylor series in a neighborhood of the point $v = v_0$.

Definition 10.2.1. *[22] The differential transformation of a real function $w(v)$ at a point $v_0 \in \mathbb{R}$ is $\mathcal{D}\{w(v)\}[v_0] = \{W(k)[v_0]\}_{k=0}^{\infty}$, where $W(k)[v_0]$, the differential transform of the k^{th} derivative of the function $w(v)$ at v_0, is defined as*

$$W(k)[v_0] = \frac{1}{k!}\left[\frac{d^k w(v)}{dv^k}\right]_{v=v_0}. \tag{10.1}$$

Definition 10.2.2. *[22] The inverse differential transformation is given by*

$$w(v) = \mathcal{D}^{-1}\Big\{ \{W(k)[v_0]\}_{k=0}^{\infty} \Big\}[v_0] = \sum_{k=0}^{\infty} W(k)[v_0](v - v_0)^k. \tag{10.2}$$

Combining Definition 10.2.1 and Definition 10.2.2 give an expression of the function w in the form of the Taylor series:

$$w(v) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{d^k w(v)}{dv^k}\right]_{v=v_0} (v - v_0)^k. \tag{10.3}$$

In practical applications, the function $w(v)$ is expressed by a finite sum

$$w(v) = \sum_{k=0}^{N} W(k)[v_0](v - v_0)^k, \tag{10.4}$$

since N can be chosen large enough to ensure that the effect of the remainder $\sum_{k=N+1}^{\infty} W(k)[v_0](v - v_0)^k$ is arbitrarily small. The results which are used in this chapter are listed in Table 10.1 without proofs.

10.2.2 Faà di Bruno's formula and Bell polynomials

In the literature, it has been observed that differential transformation is not applied directly to nonlinear terms like $w^n, n \in \mathbb{N}$ or e^w. Authors [23] used Adomian polynomials to compute the differential transform of nonlinear terms. However, the differential transformation of nonlinear terms can be determined without calculating and evaluating symbolic derivatives by applying Faà di Bruno's formula to non-linear terms.

TABLE 10.1: Formulae of the differential transform method

	Original function	Transformed function
1	$\frac{d^n w(v)}{dv^n}$	$(k+1)(k+2)(k+3)\ldots(k+n)W(k+n)$
2	$w(v)=v^n$	$\delta(k-n)=\begin{cases}1, k=n\\0, k\neq n\end{cases}$
3	$e^{\alpha v}$	$\frac{\alpha^k}{k!}$
4	$w_1(v)\,w_2(v)$	$\sum_{i=0}^{k} W_1(i)\,W_2(k-i)$
5	$w(\alpha v)$	$\alpha^k W(k)$
6	$w_1(\alpha_1 v)\,w_2(\alpha_2 v)$	$\sum_{i=0}^{k}(\alpha_1)^i(\alpha_2)^{k-i}W_1(i)\,W_2(k-i)$

Definition 10.2.3. *[10] The partial ordinary Bell polynomials are the polynomials $\hat{B}_{k,l}(\hat{x}_1,\ldots,\hat{x}_{k-l+1})$ in an infinite number of variables $\hat{x}_1,\hat{x}_2,\ldots$ defined by the series expansion*

$$\sum_{k\geq l}\hat{B}_{k,l}(\hat{x}_1,\ldots,\hat{x}_{k-l+1})v^k=\left(\sum_{m\geq 1}\hat{x}_m v^m\right)^l, l=0,1,2,\ldots. \tag{10.5}$$

Lemma 10.2.4. *[22] The partial ordinary Bell polynomials $\hat{B}_{k,l}(\hat{x}_1,\ldots,\hat{x}_{k-l+1}), l=0,1,2,\ldots,k\geq l$ satisfy the recurrence relation*

$$\hat{B}_{k,l}(\hat{x}_1,\ldots,\hat{x}_{k-l+1})=\sum_{l=1}^{k-l+1}\frac{i.l}{k}\hat{x}_i\hat{B}_{k-i,l-1}(\hat{x}_1,\ldots,\hat{x}_{k-i-l+2}), \tag{10.6}$$

where $\hat{B}_{0,0}=1$ and $\hat{B}_{k,0}=0$ for $k\geq 1$.

Theorem 10.2.5. *[22] Let g and f be real functions analytic near v_0 and $g(v_0)$, respectively, and let h be the composition $h(v)=(fog)(v)=f(g(v))$. Denote $D\{g(v)\}[v_0]=\{G(k)\}_{k=0}^{\infty}$, $D\{f(v)\}[g(v_0)]=\{F(k)\}_{k=0}^{\infty}$ and $D\{(f\circ g)(v)\}[v_0]=\{H(k)\}_{k=0}^{\infty}$ the differential transformations of functions g, f, and h at v_0, $g(v_0)$, and v_0, respectively. Then the numbers $H(k)$ in the sequence $\{H(k)\}_{k=0}^{\infty}$ satisfy the relations*

$$\begin{aligned}H(0)&=F(0)\\ H(k)&=\sum_{l=1}^{k}F(l)\,\hat{B}_{k,l}(G(1),\ldots,G(k-l+1))\ for\ k\geq 1.\end{aligned} \tag{10.7}$$

10.2.3 Description of the method

Consider the proportional delay differential equation defined by

$$w^{n}(v)=f\left(v,w(v),w'(v),\ldots,w^{(n-1)}(v),w\left(\frac{v}{\alpha_1}\right),\right.$$
$$\left.w\left(\frac{v}{\alpha_2}\right),\ldots,w\left(\frac{v}{\alpha_r}\right)\right), \tag{10.8}$$

where $\alpha_i \geq 1$ and $w^{(n)}$ is the n^{th} derivative of w with respect to v, for $n, r \in \mathbb{N}$.
Consider equation (10.8) subject to initial function $\phi(v) \in C^{n}([v^{*},0],\mathbb{R})$ where $v^{*}<0$ such that

$$\phi(v_0)=w(v_0),\ \phi'(v_0)=w'(v_0),\ldots,\phi^{(n-1)}(v_0)=w^{(n-1)}(v_0), \tag{10.9}$$

and subject to initial conditions

$$w(v_0)=w_0,\ w'(v_0)=w_1,\ldots,w^{(n-1)}(v_0)=w_{n-1}. \tag{10.10}$$

Now equation (10.8) can be written in the form

$$L(w)+R(w)+M(v)=N(w). \tag{10.11}$$

The linear terms are split into L and R, where L is the highest order bounded linear operator and R is the remaining of the linear operators which are also bounded, M are continuous known functions satisfy the Lipschitz condition, and N are nonlinear terms.
Apply DT with Bell polynomial on equation (10.10)–(10.11),

$$\mathcal{D}(L(w))+\mathcal{D}(R(w))+\mathcal{D}(M(v))=\mathcal{B}(N(w)), \tag{10.12}$$
$$\mathcal{D}(w(v_0))=w_0,\mathcal{D}(w'(v_0))=w_1,\ldots,\mathcal{D}\left(w^{(n-1)}(v_0)\right)=w_{n-1}, \tag{10.13}$$

where $\mathcal{D}$ is DT operator and $\mathcal{B}$ is Bell polynomial operator.
From equation (10.12)–(10.13) we obtain following recursive relations,

$$\frac{(k+n)!}{k!}W(k+n)+W(k)+M(k)=\mathcal{B}(N(w)), \tag{10.14}$$
$$W(0)=w_0,W(1)=w_1,\ldots,W(n-1)=\frac{1}{(n-1)!}w_{(n-1)}. \tag{10.15}$$

If $N(w)=H(v)=f(g(v))$ then nonlinear Bell polynomial operator $\mathcal{B}$ are defined by Theorem (10.2.5) as

$$H(0) = F(0),$$
$$H(k) = \sum_{l=1}^{k} F(l).\hat{B}_{k,l}(G(1),\ldots,G(k-l+1)) \; for \; k \geq 1. \tag{10.16}$$

We can easily obtain different components using equations (10.14)–(10.16), and then using inverse transformation, we obtain approximate solution in the form of Taylor series

$$w(v) = \sum_{k=0}^{\infty} W(k)(v - v_0)^k. \tag{10.17}$$

10.2.4 Convergence results

In this section, we discuss the convergence results used in this chapter. The proof is taken from [23, 27].

Theorem 10.2.6. *Let f be an analytical function in $[0,T] \times \mathbb{R}^n \times \mathbb{R}^n$. Assume that problem (10.8) has unique solution in some interval $[0,T]$. Let $B_k = W(k)v^k$. If there exist a constant δ, $0 < \delta < 1$, $k_0 \in \mathbb{N}$ such that $\|(B_{k+1}(v))\| \leq \alpha \|(B_k(v))\|$ for all, then the series converges to the unique solution on the interval $J = [0,\alpha]$, $\alpha \leq T$.*

Proof. Let $C^n(J)$ be a Banach space of vector-valued functions $h(v) = (h_1(v), \ldots, h_p(v))^T$ with continuous derivatives up to order n and norm $\|h(v)\| = \max_{i=1,\ldots,p} \max_{j=0,\ldots,n} \max_{v \in J} \left|h_i^{(j)}(v)\right|$.
Assume $S_l = \sum_{k=0}^{l} B_k(v)$. Now it is sufficient to prove that sequence $\{S_l\}$ is a Cauchy sequence in $C^n(J)$.
Due to

$$\|S_{l+1} - S_l\| = \|B_{l+1}(v)\| \leq \delta \|B_l(v)\| \leq \ldots \delta^{l-n_0+1} \|B_{n_0}(v)\|$$

for every $l, m \in \mathbb{N}$, $l \geq m > n_0$, we get

$$\begin{aligned} \|S_l - S_m\| &= \left\| \sum_{j=m}^{l-1} (S_{j+1} - S_j) \right\| \leq \sum_{j=m}^{l-1} \|(S_{j+1} - S_j\| \leq \sum_{j=m}^{l-1} \delta^{j-n_0+1} \|B_{n_0}(v)\| \\ &= \delta^{m-n_0+1}(1 + \delta + \delta^2 + \cdots + \delta^{l-m+1}) \|B_{n_0}(v)\| \\ &= \frac{1 - \delta^{l-m}}{1 - \delta} \delta^{m-n_0+1} \|B_{n_0}(v)\|. \end{aligned} \tag{10.18}$$

Since $0 < \delta < 1$, it follows that

$$\lim_{l,m \to \infty} \|S_l - S_m\| = 0.$$

Therefore $\{S_l\}$ is a Cauchy sequence in $C^n(J)$ and the proof is complete. □

Theorem 10.2.7. *Suppose that the assumptions of Theorem (10.2.6) are valid. Then for the truncated series* $\sum_{k=0}^{m} B_k(v)$, *the following error estimate holds*

$$\left\| w(v) - \sum_{k=0}^{m} B_k(v) \right\| \leq \frac{1}{1-\delta}\delta^{m-m_0+1} \max_{i=1,\ldots,p} \max_{j=0,\ldots,n} \left| \frac{m_0!}{(m_0-j)!} W_i(m_0)\alpha^{m_0-j} \right|$$

for any $m \geq 0$, $m \geq m_0$.

Proof. Without loss of generality, we can choose $m_0 \geq n$, where n is the order of the system (10.8). From inequality (10.18) we have

$$\begin{aligned} \|S_l - S_m\| &\leq \frac{1-\delta^{l-m}}{1-\delta}\delta^{m-m_0+1} \|B_{m_0}(v)\| \\ &= \frac{1-\delta^{l-m}}{1-\delta}\delta^{m-m_0+1} \max_{i=1,\ldots,p} \max_{j=0,\ldots,n} \left| \frac{m_0!}{(m_0-j)!} W_i(m_0)\alpha^{m_0-j} \right| \end{aligned} \tag{10.19}$$

for $l \geq m \geq m_0$.
From $0 < \delta < 1$ it follows $(1 - \delta^{l-m}) < 1$. Hence, inequality (10.19) can be reduced to

$$\frac{1}{1-\delta}\delta^{m-m_0+1} \max_{i=1,\ldots,p} \max_{j=0,\ldots,n} \left| \frac{m_0!}{(m_0-j)!} W_i(m_0)\alpha^{m_0-j} \right|$$

Hence, we use the fact that $l \to \infty$, $S_l \to w(v)$, and so proof is complete. □

10.2.5 Error estimate

For comparison, absolute error and maximum absolute error are computed and defined as

$$E_N(v) := |w(v) - w_N(v)|,$$
$$E_{N,\infty} := \max_{0 \leq v \leq 1} E_N(v),$$

where $w(v)$ is the exact solution and $w_N(v)$ is the truncated series solution with degree N. Furthermore, the relative error between exact and approximate solution is defined by $R_N(v) := \frac{E_N(v)}{|w(v)|}$.

10.3 Applications

Five examples are discussed to show reliability and accuracy of the presented method. The MATHEMATICA software version 11 has been used for numerical computations.

10.3.1 Example 1

Consider the following linear proportional DDE

$$w''(v) = 2e^{\frac{2v}{3}} w\left(\frac{v}{3}\right) - w'(v) - w\left(\frac{v}{2}\right) + e^{v/2},\ 0 \le v \le 1, \tag{10.20}$$

with initial conditions

$$w(0) = w'(0) = 1. \tag{10.21}$$

The exact solution is given by

$$w(v) = e^{v}. \tag{10.22}$$

Applying differential transform to equations (10.20)–(10.21), we obtain the following recursive relation

$$W(k+2) = \frac{1}{(k+1)(k+2)}\left(2\sum_{r=0}^{k}\left(\frac{1}{3}\right)^{k}\frac{2^{k-r}}{(k-r)!}W(r) - (k+1)W(k+1) - \left(\frac{1}{2}\right)^{k} W(k) + \left(\frac{1}{2}\right)^{k}\frac{1}{k!}\right), \tag{10.23}$$

$$W(0) = W(1) = 1. \tag{10.24}$$

Using equations (10.23)–(10.24), we obtain the following components,

$$k=0,\ W(2) = \frac{1}{2}(2W(0) - W(1) - W(0) + 1) = \frac{1}{2!},$$
$$k=1,\ W(3) = \frac{1}{6}\left(2\left(\frac{2}{3}W(0) + \frac{1}{3}W(1)\right) - 2W(2) - \frac{1}{2}W(1) + \frac{1}{2}\right) = \frac{1}{3!},$$
$$k=2,\ W(4) = \frac{1}{4!}, \ldots, \text{ and so on.} \tag{10.25}$$

Now, using equation (10.4), the series solution is given by

$$w(v) = 1 + v + \frac{1}{2!}v^2 + \frac{1}{3!}v^3 + \ldots, \tag{10.26}$$

which converges to the exact solution given by equation (10.22). The approximate solution for $N = 12$ is compared with the analytical solution in Table 10.2, where N represents a number of terms considered. Table 10.3 lists the maximal absolute error of approximate results obtained by the present method for $N = 5, 10$, and 15. Figure 10.1 depicts absolute errors for the numerical solutions for $N = 5$, 10, and 15. From these results, it is clear that absolute errors and maximal absolute errors all decline systematically with the increase in N.

TABLE 10.2: Comparison of numerical solution $w(v)$ with exact solution when $N = 12$ for Example 1

v	$w(v)$	$w_N(v)$	$R_N(v)$
0.1	1.105170918	1.105170918	2.0E-16
0.2	1.221402758	1.221402758	2.0E-16
0.3	1.349858807	1.349858807	1.6E-16
0.4	1.491824697	1.491824697	7.4E-16
0.5	1.648721270	1.648721270	1.2E-14
0.6	1.822118800	1.822118800	1.2E-13
0.7	2.013752707	2.013752707	8.1E-13
0.8	2.225540928	2.225540928	4.2E-12
0.9	2.459603111	2.459603111	1.7E-11
1.0	2.718281828	2.718281828	6.3E-11

TABLE 10.3: Maximum absolute errors for w of Example 1

N	$E_{N,\infty}$
5	9.9E-03
10	3.0E-07
15	8.1E-13

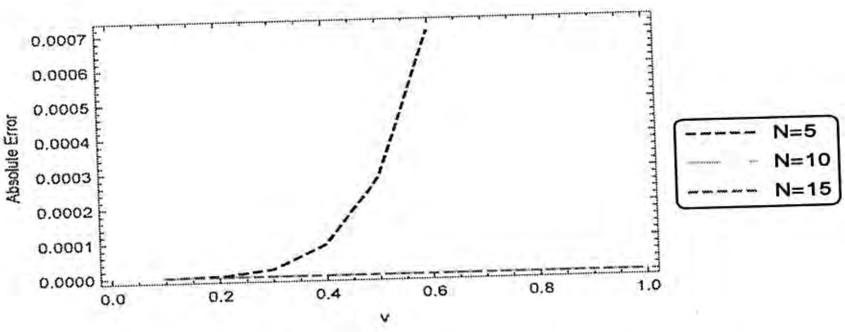


FIGURE 10.1: Absolute errors for $N = 5$, 10, and 15 of Example 1.

10.3.2 Example 2

Consider the following linear system of proportional DDEs

$$u'(v) = u(v) + 2e^{v} w\left(\frac{v}{3}\right) - e^{2v}, \tag{10.27}$$

$$w'(v) = 2w(v) + e^{2v} u\left(\frac{v}{2}\right),\ 0 \leq v \leq 1, \tag{10.28}$$

with initial conditions

$$u(0) = w(0) = 1. \tag{10.29}$$

The exact solution is given by

$$u(v) = e^{2v},\ w(v) = e^{3v}. \tag{10.30}$$

Employing the differential transform to equations (10.27)–(10.29), we obtain the following recursive relation

$$U(k+1) = \frac{1}{(k+1)}\left(U(k) + 2\sum_{r=0}^{k} \frac{1}{r!}\left(\frac{1}{3}\right)^{k-r} W(k-r) - \frac{2^k}{k!}\right), \tag{10.31}$$

$$W(k+1) = \frac{1}{(k+1)}\left(2W(k) + \sum_{r=0}^{k} \frac{2^r}{r!}\left(\frac{1}{2}\right)^{k-r} U(k-r)\right), \tag{10.32}$$

$$U(0) = W(0) = 1. \tag{10.33}$$

Using equations (10.31)–(10.33), we obtain the following components,

$$\begin{aligned}
k = 0,\ U(1) &= U(0) + 2W(0) - 1 = 2,\\
W(1) &= 2W(0) + U(0) = 3,\\
k = 1,\ U(2) &= \frac{1}{2}\left(U(1) + 2\left(\frac{1}{3}W(1) + W(0)\right) - 2\right) = \frac{2^2}{2!},\\
W(2) &= \frac{1}{2}\left(2W(1) + \frac{1}{2}U(1) + 2U(0)\right) = \frac{3^2}{2!},\\
k = 2,\ U(3) &= \frac{2^3}{3!},\\
W(3) &= \frac{3^3}{3!}, \ldots, \text{ and so on.}
\end{aligned} \tag{10.34}$$

Now, with the help of equation (10.4), the series solution is given by

$$u(v) = 1 + 2v + \frac{2^2}{2!}v^2 + \frac{2^3}{3!}v^3 + \ldots, \tag{10.35}$$

$$w(v) = 1 + 3v + \frac{3^2}{2!}v^2 + \frac{3^3}{3!}v^3 + \ldots, \tag{10.36}$$

which converges to the exact solution given by equation (10.30).

The approximate solution for $N = 12$ is compared with the analytical solution in Tables (10.4)–(10.5), where N represents a number of terms considered. Table (10.6) lists the maximal absolute error of approximate results obtained by the present method for $N = 5, 10$, and 15. Figure 10.2 depicts absolute errors for the numerical solutions for $N = 5$, 10, and 15. From these results, it is clear that absolute errors and maximal absolute errors all decline systematically with the increase in N.

TABLE 10.4: Comparison of numerical solution $u(v)$ with exact solution when $N = 12$ for Example 2

v	$u(v)$	$u_N(v)$	$R_N(v)$
0.1	1.221402758	1.221402758	0
0.2	1.491824698	1.491824698	2.4E-14
0.3	1.822118800	1.822118800	2.6E-12
0.4	2.225540928	2.225540928	6.8E-11
0.5	2.718281828	2.718281826	8.3E-10
0.6	3.320116923	3.320116902	6.1E-09
0.7	4.055199967	4.055199834	3.2E-08
0.8	4.953032424	4.953031755	1.3E-07
0.9	6.049647464	6.049644666	4.6E-07
1.0	7.389056099	7.389046016	1.3E-06

TABLE 10.5: Comparison of numerical solution $w(v)$ with exact solution when $N = 12$ for Example 2

v	$w(v)$	$w_N(v)$	$R_N(v)$
0.1	1.349858808	1.349858808	9.8E-16
0.2	1.8221188	1.8221188	2.6E-12
0.3	2.459603111	2.459603111	2.5E-10
0.4	3.320116923	3.320116902	6.1E-09
0.5	4.48168907	4.481688764	6.8E-08
0.6	6.049647464	6.049644666	4.6E-07
0.7	8.166169913	8.166151643	2.2E-06
0.8	11.02317638	11.0230832	8.4E-06
0.9	14.87973172	14.87933803	2.6E-05
1.0	20.08553692	20.08410308	7.1E-05

TABLE 10.6: Maximum absolute errors for u and w of example 2

N	$E_{N,\infty}$ **for** u	$E_{N,\infty}$ **for** w
5	3.8E-01	3.7E-00
10	3.4E-04	2.2E-02
15	2.8E-08	1.3E-05

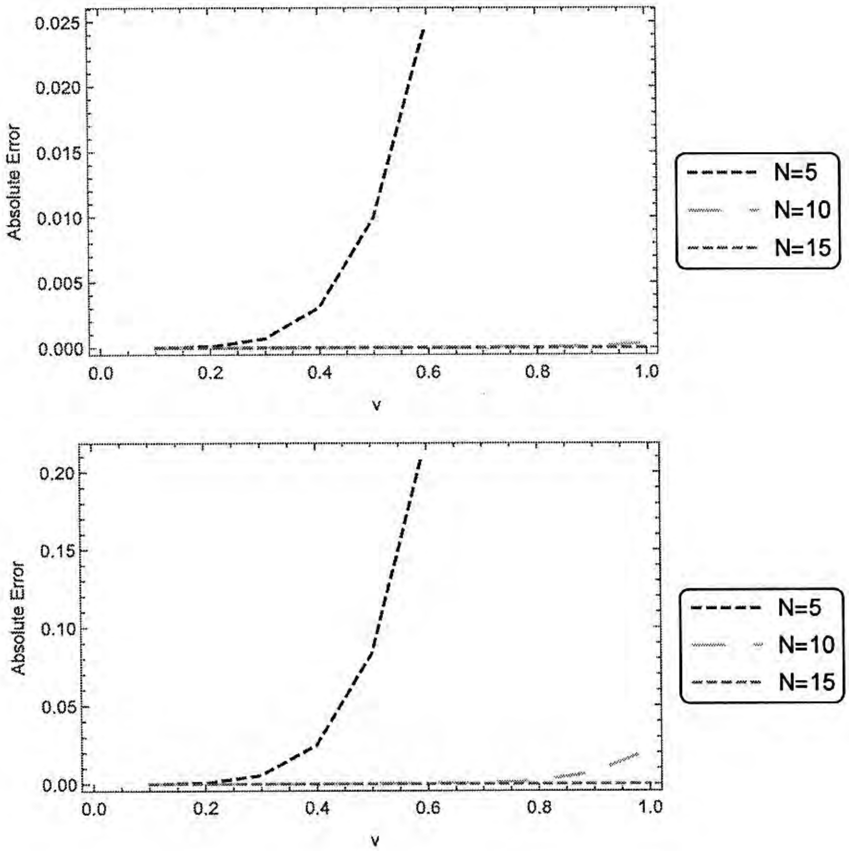


FIGURE 10.2: Absolute errors for u and w for N = 5, 10, and 15 of Example 2.

10.3.3 Example 3

Consider the following nonlinear system of proportional DDEs

$$u'(v) = u\left(\frac{v}{2}\right) w\left(\frac{v}{4}\right) - u(v) - u\left(\frac{v}{4}\right), \tag{10.37}$$

$$w'(v) = u\left(\frac{v}{2}\right) w(v) - w(v) - w\left(\frac{v}{2}\right),\ 0 \le v \le 1, \tag{10.38}$$

with initial conditions

$$u(0) = w(0) = 1. \tag{10.39}$$

The exact solution is given by

$$u(v) = e^{-v},\ w(v) = e^{v}. \tag{10.40}$$

Applying differential transform to equations (10.37)–(10.39), we obtain the following recursive relation

$$U(k+1) = \frac{1}{(k+1)}\left(\sum_{r=0}^{k}\left(\frac{1}{2}\right)^{2k-r} U(r) W(k-r) - U(k) - \left(\frac{1}{4}\right)^{k} U(k)\right), \tag{10.41}$$

$$W(k+1) = \frac{1}{(k+1)}\left(\sum_{r=0}^{k}\left(\frac{1}{2}\right)^{r} U(r) W(k-r) + W(k) - \left(\frac{1}{2}\right)^{k} W(k)\right), \tag{10.42}$$

$$U(0) = W(0) = 1. \tag{10.43}$$

Using equations (10.41)–(10.43), we obtain the following components,

$$\begin{aligned}
k = 0,\ & U(1) = U(0) W(0) - U(0) - U(0) = -1,\\
& W(1) = U(0) W(0) + W(0) - W(0) = 1,\\
k = 1,\ & U(2) = \frac{1}{2}\left(\frac{1}{4}U(0) W(1) + \frac{1}{2}U(1) W(0) - U(1) - \frac{1}{4}U(1)\right) = -\frac{1}{2!},\\
& W(2) = \frac{1}{2}\left(U(0) W(1) + \frac{1}{2}U(1) W(0) + W(1) - \frac{1}{2}U(1)\right) = \frac{1}{2!},\\
k = 2,\ & U(3) = -\frac{1}{3!},\\
& W(3) = \frac{1}{3!}, \ldots,\ \text{and so on.}
\end{aligned} \tag{10.44}$$

Now, with the help of equation (10.4), the series solution is given by

$$u(v) = 1 - v + \frac{1}{2!}v^2 - \frac{1}{3!}v^3 + \ldots, \tag{10.45}$$

$$w(v) = 1 + v + \frac{1}{2!}v^2 + \frac{1}{3!}v^3 + \ldots, \tag{10.46}$$

which converges to the exact solution given by equation (10.40).

The approximate solution for $N = 12$ is compared with the analytical solution in Tables (10.7)–(10.8), where N represents a number of terms considered. Table 10.9 lists the maximal absolute error of approximate results obtained by the present method for $N = 5$, 10, and 15. Figure 10.3 depicts absolute errors for the numerical solutions for $N = 5$, 10, and 15. From these results, it is clear that absolute errors and maximal absolute errors all decline systematically with the increase in N.

TABLE 10.7: Comparison of numerical solution $u(v)$ with exact solution when $N = 12$ for Example 3

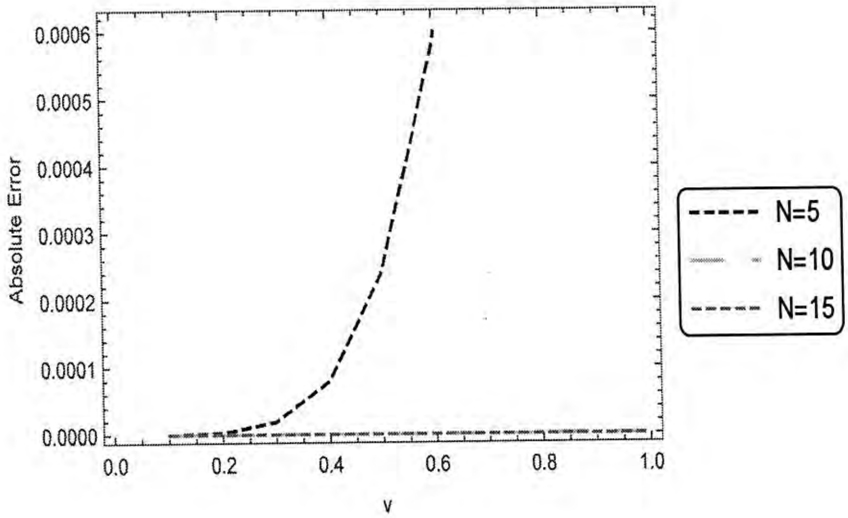
v	$u(v)$	$u_N(v)$	$R_N(v)$
0.0	1.000000000	1.000000000	0
0.1	0.904837418	0.904837418	1.2E-16
0.2	0.818730753	0.818730753	1.3E-16
0.3	0.740818220	0.740818220	1.3E-16
0.4	0.670320046	0.670320046	1.4E-15
0.5	0.606530659	0.606530659	3.1E-14
0.6	0.548811636	0.548811636	3.6E-13
0.7	0.496585303	0.496585303	2.9E-12
0.8	0.449328964	0.449328964	1.8E-11
0.9	0.406569659	0.406569659	9.4E-11
1.0	0.367879441	0.367879441	4.0E-10

TABLE 10.8: Comparison of numerical solution $w(v)$ with exact solution when $N = 12$ for Example 3

v	$w(v)$	$w_N(v)$	$R_N(v)$
0.0	1.000000000	1.000000000	0
0.1	1.105170918	1.105170918	2.0E-16
0.2	1.221402758	1.221402758	2.0E-16
0.3	1.349858807	1.349858807	1.6E-16
0.4	1.491824697	1.491824697	7.4E-16
0.5	1.648721270	1.648721270	1.2E-14
0.6	1.822118800	1.822118800	1.2E-13
0.7	2.013752707	2.013752707	8.1E-13
0.8	2.225540928	2.225540928	4.2E-12
0.9	2.459603111	2.459603111	1.7E-11
1.0	2.718281828	2.718281828	6.3E-11

TABLE 10.9: Maximum absolute errors for u and w of Example 3

N	$E_{N,\infty}$ **for** u	$E_{N,\infty}$ **for** w
5	1.2E-03	9.9E-03
10	2.3E-08	3.0E-07
15	7.1E-13	8.1E-13



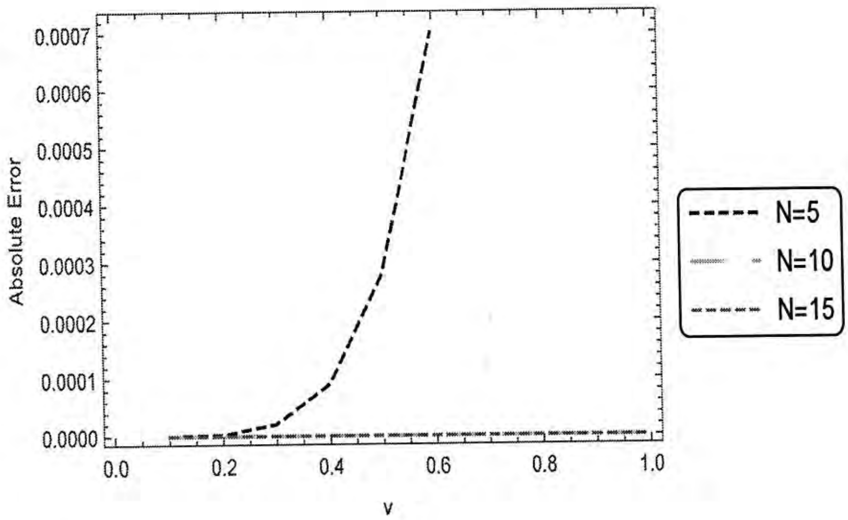


FIGURE 10.3: Absolute errors for u and w for N = 5, 10, and 15 of Example 3.

10.3.4 Example 4

Consider the following nonlinear proportional DDE

$$(1+v)\,w'(v) = (1+v)\,e^{w\left(\frac{v}{2}\right)} - \frac{1}{2}v^2 - \frac{3}{2}v,\ 0 \le v \le 1, \tag{10.47}$$

with initial condition

$$w(0) = 0. \tag{10.48}$$

The exact solution is given by

$$w(v) = \log(1+v). \tag{10.49}$$

Denote $h(v) = f(g(v))$, where $g(v) = w\left(\frac{v}{2}\right)$ and $f(x) = e^x$.
Differential transform of $f(x)$ is represented by $F(k)$ then

$$F(k) = \frac{1}{k!}, \tag{10.50}$$

and differential transform of $h(v)$ is represented by $H(k)$ then using theorem (10.2.5), we have $H(0) = 1$ and

$$H(k) = \left(\frac{1}{2}\right)^k \sum_{l=1}^{k} F(l)\,\hat{B}_{k,l}(W(1), \ldots, W(k-l+1)) \text{ for } k \ge 1.$$

Applying differential transform to equations (10.47)–(10.48), we obtain the following recursive relation

$$\begin{aligned} W(k+1) &= \frac{1}{(k+1)}\Bigg(-\sum_{r=0}^{k}\delta(r-1)(k-r+1)\,W(k-r+1) + H(k) \\ &\quad + \sum_{r=0}^{k}\delta(r-1)\,H(k-r) - \frac{1}{2}\delta(k-2) - \frac{3}{2}\delta(k-1)\Bigg) \quad (10.51) \\ W(0) &= 0. \quad (10.52) \end{aligned}$$

Using equations (10.50)–(10.52) we obtain following components,

$$\begin{aligned} k=0,\ W(1) &= H(0) = 1, \\ k=1,\ H(1) &= \left(\frac{1}{2}\right) F(1)\,\hat{B}_{1,1}(W(1)) = \frac{1}{2}F(1)\,W(1) = \frac{1}{2}, \\ W(2) &= \frac{1}{2}\left(-W(1) + H(1) + H(0) - \frac{3}{2}\right) = -\frac{1}{2}, \\ k=2,\ H(2) &= \left(\frac{1}{2}\right)^2\left(\sum_{l=1}^{k} F(l)\,\hat{B}_{2,1}(W(1), W(2))\right) \end{aligned}$$

$$= \frac{1}{4}\left(F(1)W(2)+F(2)W^2(1)\right)=0,$$

$$W(3) = \frac{1}{3}\left(-2W(2)+H(2)+H(1)-\frac{1}{2}\right)$$

$$= \frac{1}{3},\dots \text{ and so on.} \tag{10.53}$$

Now, with the help of equation (10.4), the series solution is given by

$$w(v) = v - \frac{1}{2}v^2 + \frac{1}{3}v^3 - \dots, \tag{10.54}$$

which converges to the exact solution given by equation (10.49).

The approximate solution for $N = 15$ is compared with the analytical solution in Table 10.10, where N represents a number of terms considered. Table 10.11 lists the maximal absolute error of approximate results obtained by the present method for $N = 5$, 10, and 15. Figure 10.4 depicts absolute errors for the numerical solutions for $N = 5$, 10, and 15.

TABLE 10.10: Comparison of numerical solution $w(v)$ with exact solution when $N = 15$ for Example 4

v	$w(v)$	$w_N(v)$	$R_N(v)$
0.1	0.0953101798	0.0953101798	7.2E-16
0.2	0.1823215568	0.1823215568	1.8E-12
0.3	0.2623642645	0.2623642647	7.9E-10
0.4	0.3364722366	0.3364722561	5.7E-08
0.5	0.4054651081	0.4054657568	1.5E-06
0.6	0.4700036292	0.4700149031	3.9E-05
0.7	0.5306282511	0.5307535353	2.3E-04
0.8	0.5877866649	0.5887909192	1.7E-03
0.9	0.6418538862	0.6481288691	9.7E-03
1.0	0.6931471806	0.7253718504	4.6E-02

TABLE 10.11: Maximum absolute errors for w of Example 4

N	$E_{N,\infty}$
5	9.0E-02
10	4.7E-02
15	3.2E-02

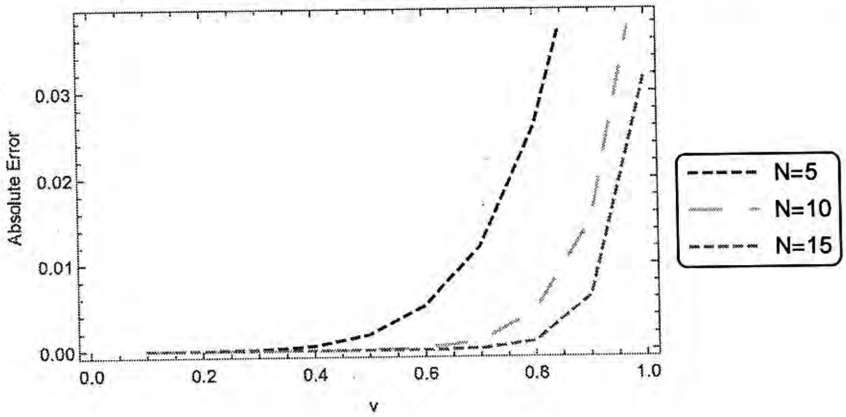


FIGURE 10.4: Absolute errors for $N = 5$, 10, and 15 of Example 4.

10.3.5 Example 5

Consider the following nonlinear proportional DDE

$$w'(v) = 4w'\left(\frac{v}{2}\right)\sqrt{1 - w^2\left(\frac{v}{2}\right)} - 2,\ 0 \le v \le 1, \tag{10.55}$$

with initial condition

$$w(0) = 0. \tag{10.56}$$

The exact solution is given by

$$w(v) = \sin(2v). \tag{10.57}$$

Denote $h(v) = f(g(v))$, where $g(v) = w\left(\frac{v}{2}\right)$ and $f(x) = \sqrt{1 - x^2}$. Differential transform of $f(x)$ is represented by $F(k)$ then

$$F(k) = \begin{cases} \binom{1/2}{k}(-1)^k, & \text{k is even} \\ 0, & \text{k is odd,} \end{cases} \tag{10.58}$$

and differential transform of $h(t)$ is represented by $H(k)$ then using theorem (10.2.5), we have $H(0) = 1$ and

$$H(k) = \left(\frac{1}{2}\right)^k \sum_{l=1}^{k} F(l)\hat{B}_{k,l}(W(1), \ldots, W(k-l+1)) \text{ for } k \ge 1.$$

Applying differential transform to equations (10.55)–(10.56), we obtain the following recursive relation

$$W(k+1) = \frac{4}{(k+1)}\left(\sum_{r=0}^{k}\left(\frac{1}{2}\right)^{k-r+1}(k-r+1)H(r)W(k-r+1) - 2\delta(k)\right) \tag{10.59}$$

$$W(0) = 0. \tag{10.60}$$

Using equations (10.58)–(10.60) we obtain following components,

$$
\begin{aligned}
k=0,\ W(1) &= \frac{4}{2}H(0)W(1)-2=2,\\
k=1,\ H(1) &= \left(\frac{1}{2}\right)F(1)\hat{B}_{1,1}(W(1))=\frac{1}{2}F(1)W(1)=0,\\
W(2) &= \frac{4}{2}\left(\frac{2}{4}H(0)W(2)+\frac{1}{2}H(1)W(1)\right)=0,\\
k=2,\ H(2) &= \left(\frac{1}{2}\right)^2\left(\sum_{l=1}^{k}F(l)\hat{B}_{2,1}(W(1),W(2))\right)\\
&= \frac{1}{4}\left(F(1)W(2)+F(2)W^2(1)\right)=-\frac{1}{2},\\
W(3) &= \frac{4}{3}\left(\frac{3}{8}H(0)W(3)+\frac{2}{4}H(1)W(2)+\frac{1}{2}H(2)W(1)\right)\\
&= -\frac{4}{3},\ldots \text{ and so on.}
\end{aligned}
\tag{10.61}
$$

Now, with the help of equation (10.4), the series solution is given by

$$
w(v)=2v-\frac{4}{3}v^3+\ldots, \tag{10.62}
$$

which converges to the exact solution given by equation (10.57).

The approximate solution for $N=12$ is compared with the analytical solution in Table (10.12), where N represents a number of terms considered. Table (10.13) lists the maximal absolute error of approximate results obtained by the present method for $N=5,10$, and 15. Figure 10.5 depicts absolute errors for the numerical solutions for $N=5,10$, and 15. From these results, it

TABLE 10.12: Comparison of numerical solution $w(v)$ with exact solution when $N=12$ for Example 5

v	$w(v)$	$w_N(v)$	$R_N(v)$
0.1	0.1986693308	0.1986693308	7.2E-16
0.2	0.3894183423	0.3894183423	1.3E-16
0.3	0.5646424734	0.5646424734	2.8E-15
0.4	0.7173560909	0.7173560909	3.7E-13
0.5	0.8414709848	0.8414709846	1.2E-11
0.6	0.9320390860	0.9320390843	1.8E-09
0.7	0.9854497300	0.9854497174	1.2E-08
0.8	0.9995736030	0.9995735316	7.1E-08
0.9	0.9738476309	0.9738473016	3.3E-07
1.0	0.9092974268	0.9092961360	1.4E-02

TABLE 10.13: Maximum absolute errors for w of Example 5

N	$E_{N,\infty}$
5	2.4E-02
10	5.0E-05
15	3.6E-10

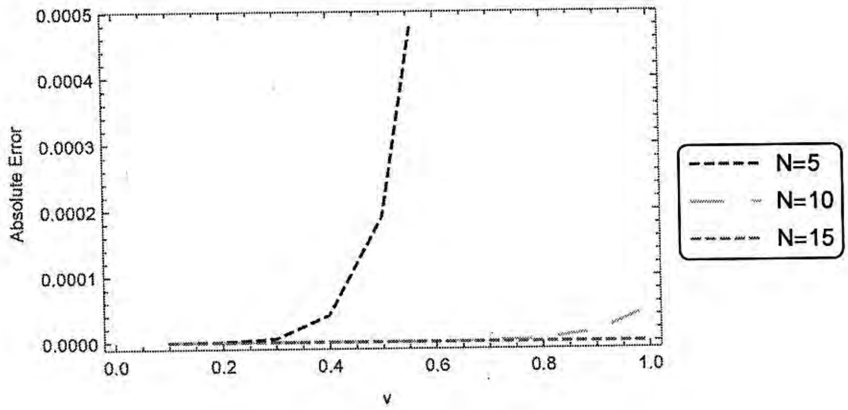


FIGURE 10.5: Absolute errors for $N = 5$, 10, and 15 of Example 5.

is clear that absolute errors and maximal absolute errors all decline systematically with the increase in N.

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