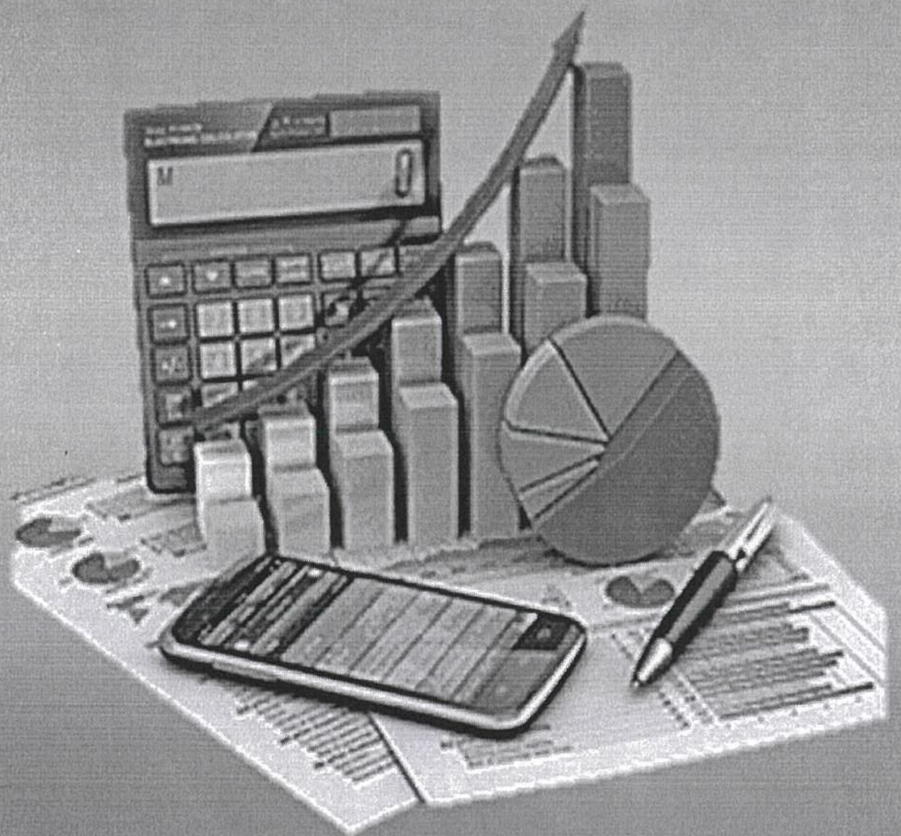
STATISTICS AND DATA SCIENCE



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A CONWAY-MAXWELL-BINOMIAL DISTRIBUTION FOR MODELING CLUSTER BINARY RESPONSES

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Abstract:

In cluster binary response analysis, the Binomial model is not suitable due to the interdependence of the data. In such cases, the Beta-Binomial (BB) model is recommended. An alternative to the BB model is the Conway-Maxwell-Binomial (CMB) distribution. This article focuses on the maximum likelihood estimation of the parameters of the CMB model . There is no analytic solution for likelihood equations, so using a self-written R program based on NR method is used to estimate MLEs iteratively. A simulation study has been conducted to see the behavior of the MLEs.

Keywords: Conway-Maxwell- Binomial distribution, Binomial model, MLEs , NR method.

Introduction:

Cluster binary response refers to a type of data that consists of binary outcomes (i.e., either 0 or 1) for multiple observations within a cluster or group. The observations within a cluster are often dependent or correlated, and as a result, traditional binary response models like the Binomial distribution may not be appropriate for analyzing this type of data. In such cases, alternative models like the Beta-Binomial is often used to capture the dependence structure among the observations within a cluster.

In toxicological studies, the number of occurrences of a certain kind of event in a litter of fetuses which may be death, malformation or mental disorder is recorded. In this example, outcome of an experiment is binary in nature. Mainly, the occurrence or non-occurrence of an event where event can be death of fetus or occurrence of certain malformation in fetus and response of interest is total count of such events. The response variable can be expressed as the sum of Bernoulli random variables for each object under study. To establish this in notation, suppose there are k litters or groups having mi number of fetuses or objects in i^{th} group. Define i = 1, 2, · · · , k; and j = 1, 2, · · · , m_i.

$$X_{ij} = \begin{cases} 1\,; if j^{th} subjectshowstheocuuranceofaneventini^{th} group. \\ 0\,; otherwise \end{cases}$$

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Thus $Y_i = \sum_{j=1}^{m_i} X_{ij}$ represents the total number of occurrences of an event in i^{th} group.

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Application of CMB distribution in cluster binary response data can be found in various fields, for e.g, In clinical trials, cluster binary response is observed when track the success or failure of a medical treatment on different patient groups is considered.

In this paper I consider a generalization of the Binomial distribution to a two-parameter distribution which is known as Conway-Maxwell-Binomial (CMB) distribution [2]

The CMB distribution consists of an extra parameter, which we denote by ϑ, and which governs the rate of decay of successive ratios of probabilities such that $P(Y = y - 1)/P(Y = y) = \frac{1}{\theta}\left[\frac{y}{m-y+1}\right]^{\vartheta}$. The CMB distribution is appealing from a theoretical point of view since it belongs to the exponential family as well as to the two-parameter power series of distributions family. As such, it allows for sufficient statistics and other properties to be elegantly derived [2].

The parameter estimation of the CMB model using the maximum likelihood method brings some challenges, since there are not explicit solutions for the maximum likelihood estimation (MLE) and it is necessary to use iteration methods. Thus, the main objective of this study is to present to estimate the parameters of the CMB model. A simulation study has been conducted to see the behavior of the MLEs.

The CMB distribution and its properties:

A random variable Y is said to follow binomial distribution if assumes only non- negative values and its probability mass function is

$$P(Y = y) = \begin{cases} \binom{m_i}{y} p^y (1-p)^{m_i - y} & ; y = 0,1,2,\ldots\ldots m_i ; 0 < p < 1 ; \text{i}= 1, 2,\ldots.\text{k} \\ 0 \quad ; & Otherwise \end{cases} \quad (1)$$

The Conway- Maxwell- Binomial (CMB) distribution is a convenient two parameter family that generalize the binomial distribution and models both positive and negative association among the Bernoulli r.vs.

The probability mass function (pmf) of the CMB distribution [3] is denoted by $Y \sim CMB(m_i, p, \vartheta)$ and given by

$$P(Y = y) = \begin{cases} \dfrac{\binom{m_i}{y}^{\vartheta} p^y (1-p)^{m_i - y}}{Z(p, \vartheta)} & ; y = 0,1,2,\ldots\ldots m_i ; 0 < p < 1; -\infty \leq \vartheta \leq \infty \\ 0 \quad ; & Otherwise \end{cases} \quad (2)$$

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Where $Z(p,\vartheta) = \sum_{k=0}^{m_i} \binom{m_i}{k}^{\vartheta} p^k (1-p)^{m_i-k}$

In particular , when $\vartheta = 1$, the pmf in equation (2) reduces to the binomial distribution, when $\vartheta > 1$, it represent under-dispersion and when $\vartheta < 1$ over-dispersion with respect to the binomial distribution.

The CMB distribution can be expressed as a sum of equi-correlated Bernoulli random variables [3] . The Compossion distribution [1] is approximated to the CMB distribution when m is getting large.

The following figure 1 presents the pmf of the CMB distribution for m = 5 and different values of p and ϑ.

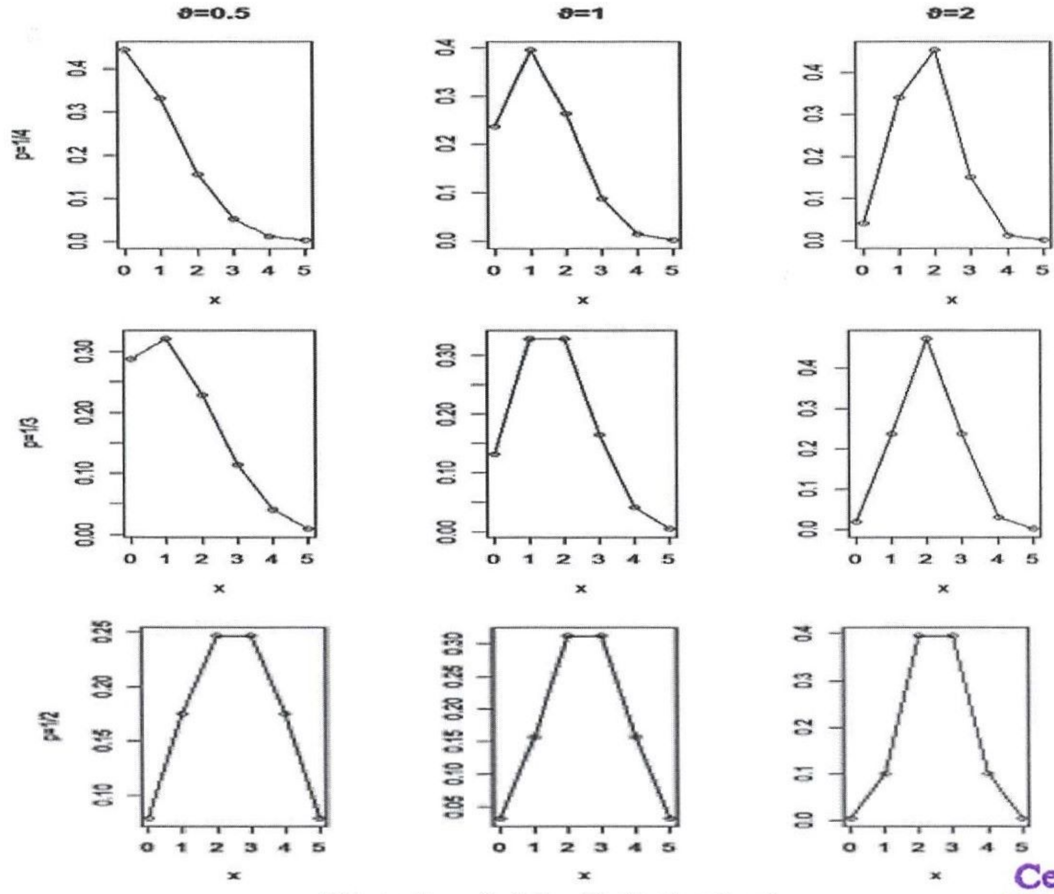

Figure 1 pmf of the CMB distribution

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Considering the reparameterization, $\theta = \frac{p}{1-p}$, the pmf of the CMB distribution is given by

$$P(Y = y) = \begin{cases} \frac{1}{Z(\theta,\vartheta)} \frac{\theta^y}{[y!(m_i - y)!]^\vartheta} & ; y = 0,1,2, \ldots .. m_i ; \theta > 0; -\infty \le \vartheta \le \infty \\ 0 \quad ; & Otherwise \end{cases} \quad (3)$$

Where $Q_i = Z(\theta, \vartheta) = \sum_{k=0}^{m_i} \frac{\theta^k}{[k!(m_i-k)!]^\vartheta}$

Maximum Likelihood Estimation of the Parameters:

Let ($Y_1, Y_2 \ldots\ldots Y_f$), be independent vectors, where each vector has exchangeable binary components.

The Likelihood function $L = L(\theta, \vartheta \,|y_1, y_2, \ldots . y_f)$ will be

$$L = \prod_{i=1}^{f} P(Y = y_i)$$

$$= \prod_{i=1}^{f} \frac{1}{Q_i} \frac{\theta^{y_i}}{[y_i!(m_i - y_i)!]^\vartheta}$$

The log- likelihood function is

$$l = \sum_{i=1}^{f} \{ y_i \log \theta - \vartheta \log[y_i! \, (m_i - y_i)!] - \log Q_i \} \text{----------------}(4)$$

The partial derivatives are given by

$$\frac{\partial l}{\partial \theta} = \sum_{i=1}^{f} \left\{ \frac{y_i}{\theta} - \left(\frac{\sum_{k=1}^{m_i} \frac{k\theta^k}{[k!(m_i-k)!]^\vartheta}}{Q_i} \right) \right\} \quad (5)$$

$$\frac{\partial l}{\partial \vartheta} = \sum_{i=1}^{f} \left\{ -\log[y_i \,! \,(m_i - y_i)!] - \left(\frac{\sum_{k=0}^{m_i} \frac{\theta^k \log[k!(m_i-k)!]}{[k!(m_i-k)!]^\vartheta}}{Q_i} \right) \right\} \quad (6)$$

Solving the equations (5) and (6) simultaneously, mle of θ and ϑ can be obtained.

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However, these likelihood equations cannot be solved analytically. Hence an iterative method such as Newton-Raphson (NR) iterative method, has to be used to solve likelihood equations. I have written R program to solve the equations (5) and (6) using NR method.

Simulation Study:

In this section a simulation study has been conducted to see the performance of the estimated parameters. Here, we have generated random observations from CMB with different cluster sizes K = (20, 50, 80, 150, 200) and sample sizes m_i = 1, 2, 3, 4, 6 respectively with different combination of true values of parameters θ andϑ.By using inverse transform technique observation from CMB distribution are generated. Finally, MLEs are computed based on 1000 iterations using self-written R program. Bias and MSE of the parameters given in the Table 1, they are computed using the following formulae.

$$\text{Bias}(\hat{\theta}_1) = E(\hat{\theta}_1) - \theta_1$$

$$\text{MSE}(\hat{\theta}_1) = E(\hat{\theta}_1 - \theta_1)^2$$

$\hat{\theta}_1 = (\hat{\theta}, \hat{\vartheta})$ is estimated parameter and $\theta_1 = (\theta, \vartheta)$ is true parameter .

Here; R = Number of replications = 1000

Table 1. Results of Simulation

	θ	0.5	0.6	0.7	0.8
ϑ					
0.1	Bias $(\hat{\theta})$	-0.001582	-0.002726	0.001176	-0.002933
	MSE$(\hat{\theta})$	0.001306	0.000886	0.000819	0.000781
	Bias $(\hat{\vartheta})$	0.018667	0.003190	0.001036	0.004220
	MSE $(\hat{\vartheta})$	0.007497	0.005779	0.003995	0.003396
0.2	Bias $(\hat{\theta})$	0.000097	-0.008027	-0.003478	-0.000092
	MSE$(\hat{\theta})$	0.001169	0.000965	0.000942	0.001044
	Bias $(\hat{\vartheta})$	-0.010122	0.022223	0.004813	0.002248
	MSE $(\hat{\vartheta})$	0.005787	0.005399	0.005244	0.003760
0.3	Bias $(\hat{\theta})$	-0.003050	-0.005379	0.004778	-0.001319

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	MSE($\hat{\theta}$)	0.001092	0.000868	0.001252	0.001293
	Bias ($\hat{\vartheta}$)	0.003555	0.001344	-0.007627	-0.002689
	MSE ($\hat{\vartheta}$)	0.007814	0.005436	0.005156	0.003559
0.4	Bias ($\hat{\theta}$)	0.004792	-0.000239	-0.001968	0.003344
	MSE($\hat{\theta}$)	0.001161	0.001079	0.000778	0.001069
	Bias ($\hat{\vartheta}$)	-0.000503	0.013884	0.024210	0.000919
	MSE ($\hat{\vartheta}$)	0.005840	0.004409	0.005784	0.004501
0.5	Bias ($\hat{\theta}$)	0.001287	-0.004993	0.002772	0.003979
	MSE($\hat{\theta}$)	0.001187	0.001363	0.000955	0.001239
	Bias ($\hat{\vartheta}$)	0.007625	0.014888	0.000682	0.004635
	MSE ($\hat{\vartheta}$)	0.009036	0.005867	0.005870	0.004729

From the table we can observed that estimated parameters are quite closer to true parameter and MSE is much smaller.

Conclusion and FutureWork:

In this study, the Conway-Maxwell-Binomial (CMB) distribution is applied in cluster binary data. The parameters are estimated using the method of maximum likelihood estimators. A simulation study has been conducted to see the behavior of the MLEs.

A future prospect of this study, the appropriateness of the fitting distribution is carried out based on the goodness of fit test and some information criteria and applies to the real-life data set and compare with other appropriate distribution.

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CERTIFICATE

This is to certify that, Prof./Dr./Mr./Ms. Rahul Ramlalit Tiwari

of RJ college of Arts & science.

has Participated/Presented a Paper (Oral / Poster) entitled A Conway - Maxwell - Binomial distribution for Modeling Cluster Binary Responses.

in **International Conference on Statistics and Data Science** (**SDS-2023**) organized by Dept. of Statistics, Sadguru Gadage Maharaj College, Karad on 24th and 25th February 2023.

S.P. Patil		
Dr. Mrs. S. P. Patil	Mrs. S. V. Mahajan	Prin. Dr. M. M. Rajmane
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