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Qualitative Theory of Dynamical Systems (2023) 22:97
https://doi.org/10.1007/s12346-023-00784-1

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Existence of Solutions for Two-Point Integral Boundary Value Problems with Impulses

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Received: 14 January 2023 / Accepted: 28 March 2023

Abstract
In this paper, we investigate the existence of at least one solution and at least two nonnegative solutions of impulsive differential equations with the two-point integral boundary conditions. We employ the recent fixed point theorems for the sum of two operators on Banach spaces. The applicability of the results is illustrated through an example.

Keywords Initial-boundary value problem · Impulses · Existence · Nonnegative solutions · Multiplicity of solutions · Fixed point · Cone · Sum of operators

Mathematics Subject Classification 34B10 · 34B37 · 47H10

1 Introduction

The theory of conventional differential equations is, no doubt, one of the versatile tools to model and study various real-world physical phenomena. There are numerous evolution processes that experience perturbations and undergo sudden changes in their states. These changes are relatively short-time as compared to the overall duration of the whole process. Such processes can be found in various fields of science and technology. For the description of applications in biology, medicine, population

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dynamics, economics, neural networks, readers can refer [23]. It is seen that conventional differential equations are inadequate to describe such phenomena. Therefore, it is natural to consider the governing differential equations along with their impulse effects. These equations are known as impulsive differential equations. Inspired by numerous applications, the theory of impulsive differential equations has been studied intensively and over the last three decades, it has been an active research area producing an extensive portfolio of results. This can be witnessed by the following published works by different authors [2–4, 13, 14, 21–23]. It is not an overemphasize that this theory is much richer than the corresponding differential equations. In consequence, it creates an important branch of nonlinear analysis.

On the other hand, due to its wide applicability in many actual phenomena, it is essential to include suitable conditions with differential equations. This often leads to the study of the initial and boundary value problems together with integral equations. Such problems form the basis of mathematical modelling of several dynamic phenomena. When the boundary of the physical process is not available for measurements, nonlocal conditions in a multi-point form may be imposed as an additional information, sufficient for the solvability. Keeping this in mind, equations with multi-point integral boundary conditions play an important and special role. They include two, three, multi-point, and nonlocal boundary conditions as special cases. Various differential equations with several types of nonlocal conditions have been studied extensively and a large number of papers are devoted on this study, see [5, 6, 10–12, 18–20, 24] to mention a few.

Ashyralyev and Sharifov [1] studied the existence and uniqueness of solution for the system of nonlinear differential equations of the type

$$x'(t) = f(t, x(t)), \quad t \in [0, T], \quad t \neq t_k, \quad k = 1, 2, 3, \ldots, p, \tag{1.1}$$

subject to impulsive conditions

$$x(t_k^+) - x(t_k) = I_k(x(t_k)), \quad k = 1, 2, 3, \ldots, p, \quad t \in [0, T], \tag{1.2}$$

and two-point integral boundary condition

$$Ax(0) + Bx(T) = \int_0^T g(s, x(s))ds, \tag{1.3}$$

where $0 < t_1 < t_2 < \ldots < t_p < t_{p+1} = T$, $p \in \mathbb{N}$, $A, B \in \mathbb{R}^{n \times n}$ are given matrices such that $\det(A+B) \neq 0$, and $f, g : [0, T] \times \mathbb{R}^n$ and $I_k : \mathbb{R}^n \to \mathbb{R}^n$ are given functions which satisfy certain conditions. A result concerning the existence of unique solution of (1.1)–(1.3) is obtained using the Banach fixed point theorem while two separate results concerning existence of at least one solution are obtained by using the Schaefer fixed point theorem and the Leray–Schauder type nonlinear alternative, respectively.

Mardanov et al. [16] also studied the existence and uniqueness of solution for the system of nonlinear differential equations of the type

$$x'(t) = f(t, x(t)), \quad t \in [0, T], \tag{1.4}$$

with two-point integral boundary conditions of the form

$$Ax(0) + \int_0^T m(s)x(s)ds + Bx(T) = \int_0^T g(s, x(s))ds, \tag{1.5}$$

where $A, B \in \mathbb{R}^{n\times n}$ are given matrices such that $\det(A + \int_0^T m(s)ds B) \neq 0$, and $m, f, g\colon [0, T] \times \mathbb{R}^n$ are given functions which satisfy certain conditions. Mardanov et al. [16] used the same tools and obtained the similar results of [1].

In this paper, we investigate for the existence of at least one solution and at least two nonnegative solutions of the following boundary value problem (BVP)

$$\begin{aligned} x'(t) &= f(t, x(t)), \quad t \in [0, T], \quad t \neq t_k,\\ \Delta x(t_k) &= I_k(x(t_k)), \quad k = 1, 2, 3, \dots, p,\\ Ax(0) + \int_0^T h(s)x(s)ds + Bx(T) &= \int_0^T g(s, x(s))ds, \end{aligned} \tag{1.6}$$

where $0 < t_1 < t_2 < \ldots < t_p < T$, $p \in \mathbb{N}$, $\Delta x(t_k) = x(t_k^+) - x(t_k)$, $k \in \{1, \dots, p\}$, $x(t_k^+)$ denotes the right limit of $x(t)$ at $t = t_k$, $k \in \{1, \dots, p\}$, and the nonlinear functions involved satisfy the following hypotheses.

(H$_1$) $f\colon [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous function, $f := (f_1, \dots, f_n)$ such that

$$|f_j(t, v)| \leq a_{1j}(t) + a_{2j}(t)|v|^{p_j}, \quad (t, v) \in [0, T] \times \mathbb{R}^n,$$

where $v := (v_1, \dots, v_n)$, $|v| = \sqrt{v_1^2 + \cdots + v_n^2}$, $p_j \geq 0$, and $a_{1j}, a_{2j}\colon [0, T] \to [0, \infty)$ are continuous functions such that $0 \leq a_{1j}, a_{2j} \leq D$, $j \in \{1, \dots, n\}$, for some positive constant D.

(H$_2$) $I_k\colon \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions, $I_k := (I_{k1}, \dots, I_{kn})$ such that

$$|I_{kj}(v)| \leq a_{1kj} + a_{2kj}|v|^{q_j}, \quad v \in \mathbb{R}^n,$$

where $q_j \geq 0$, a_{1kj} and a_{2kj} are constants such that $0 \leq a_{1kj}, a_{2kj} \leq D$, $k \in \{1, \dots, p\}$, $j \in \{1, \dots, n\}$.

(H$_3$) $h\colon [0, T] \to \mathbb{R}$ is a continuous function such that

$$|h(t)| \leq D, \quad t \in [0, T].$$

(H$_4$) $g : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, $g := (g_1, \dots, g_n)$ such that

$$|g_j(t, v)| \leq b_{1j}(t) + b_{2j}(t)|v|^{r_j}, \quad (t, v) \in [0, T] \times \mathbb{R}^n,$$

where $r_j \geq 0$ and $b_{1j}, b_{2j}\colon [0, T] \to [0, \infty)$ are continuous functions such that $0 < b_{1j}, b_{2j} \leq D$, $j \in \{1, \dots, n\}$.

(H_5) A and B are constant $n \times n$ matrices such that

$$\det\left(A + e^{-DT}B\right) \neq 0.$$

Mardanov and Sharofov [15] investigated BVP (1.6) for the existence of unique solution and of at least one solution in the following cases.

$$\begin{aligned} |f(t,u) - f(t,v)| &\leq N(t)|u-v| \text{ and} \\ |g(t,u) - g(t,v)| &\leq M(t)|u-v|, \quad t \in [0,T], \quad u, v \in \mathbb{R}^n, \end{aligned} \tag{1.7}$$

where $M, N\colon [0,T] \to [0,\infty)$ are continuous functions, and

$$\begin{aligned} |f(t,u)| &\leq N_1 \text{ and} \\ |g(t,u)| &\leq N_2, \quad t \in [0,T], \quad u \in \mathbb{R}^n, \end{aligned} \tag{1.8}$$

where N_1 and N_2 are positive constants. Note that if

$$f_j(t,u) = g_j(t,u) = u_j^2, \quad (t,u) \in [0,T] \times \mathbb{R}^n, \quad j \in \{1,\ldots,n\},$$

then f and g do not satisfy (1.7) and (1.8). Thus, the results in this paper can be considered as complimentary results to the results in [15].

The plan of this paper is as follows. In the next section, we recall some notations, definitions, and auxiliary results that we need throughout this paper. In Sect. 3, we prove our main results about the existence and multiplicity of solutions for the problem (1.6) by using recent fixed point theorems for the sum of two operators $T + S$ on Banach spaces, firstly by considering T linear and $(I - S)$ be compact, secondly by taking this sum such that T is an expansive operator and S is completely continuous one. An example is given in Sect. 4 in order to illustrate our obtained results. In Sect. 5, a concluding remarks are given.

2 Preliminary Results

In this section, we will give some preliminary material needed to prove our main results. First, we recall some notations and definitions that we need throughout this paper.

For $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ and a constant $a \in \mathbb{R}$, when we write $v \geq 0$ we have in mind $v_j \geq 0$, $j \in \{1,\ldots,n\}$, and when we write $v + a$ we have in mind $(v_1 + a, v_2, \ldots, v_n + a)$. For a matrix $C = (c_{ij})_{n\times n}$, denote $\|C\| = \max_{i,j\in\{1,\ldots,n\}} |c_{ij}|$. Let

$$PC([0,T],\mathbb{R}) = \left\{u\colon [0,T] \to \mathbb{R}\colon u \text{ is continuous for any } t \in [0,T] \setminus \cup_{k=1}^{p}\{t_k\} \text{ and } u(t_k^+) \text{ exists}, k \in \{1,2,\ldots,p\}\right\}$$

be endowed with the norm

$$\|u\|_1 = \sup_{t \in [0,T]} |u(t)|.$$

Let $E := PC([0, T], \mathbb{R})^n$ be endowed with the norm

$$\|v\| = \max_{j \in \{1,\dots,n\}} \|v_j\|_1, \quad v \in E.$$

To prove the existence of at least one solution to BVP (1.6), we will use the following fixed point theorem for a sum of two operators.

Theorem 2.1 [8, Theorem 2.1]. *Let E be an infinite dimensional Banach space and $X := \{u \in E : \|u\| \le \rho\}$, where $\rho > 0$. Also, let $T : X \to E$ be defined by $T[u] := -\varepsilon u$ for $u \in X$ and $\varepsilon > 0$ and $S : X \to E$ be continuous such that $(I - S)(X)$ resides in a compact subset of E and*

$$\{u \in E : u = \lambda(I - S)[u],\ \|u\| = \rho\} = \emptyset \text{ for any } \lambda \in \left(0, \frac{1}{\varepsilon}\right). \tag{2.1}$$

Then, there exists $u^* \in X$ so that

$$T[u^*] + S[u^*] = u^*.$$

Here $\mu X := \{\mu u : u \in X\}$ for any $\mu \in \mathbb{R}$.

Theorem 2.1 will be used to prove Theorem 3.8 and its proof can be found in [8]. In the sequel, we are concerned with the existence of multiple nonnegative fixed points for the sum of an expansive mapping and a completely continuous one. So let us recall the definitions from the available literature.

Definition 2.2 [9, A.8 in §1]. Let X and Y be real Banach spaces. A mapping $T : X \to Y$ is said to be expansive if there exists a constant $h > 1$ such that

$$\|T[u] - T[v]\|_Y \ge h\|u - v\|_X$$

for any $u, v \in X$.

Definition 2.3 [9, Definition 1.1 in §6]. Let E be a real Banach space. A mapping $S : E \to E$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Definition 2.4 [9, Definition 7.7 in §12]. A closed convex set $\mathcal{P}$ in a real Banach space E is said to be cone if

1. $\alpha u \in \mathcal{P}$ for any $\alpha \ge 0$ and for any $u \in \mathcal{P}$,
2. $u, -u \in \mathcal{P}$ implies $u = 0$.

Remark 2.5 Every cone $\mathcal{P}$ defines a partial ordering $\leq$ in E defined by

$$u \leq v \text{ if and only if } v - u \in \mathcal{P}.$$

In the sequel, $\mathcal{P}$ will refer to a cone in a Banach space $(E, \|.\|)$, Ω is a subset of $\mathcal{P}$, U is a bounded open subset of $\mathcal{P}$, and $\mathcal{P}^* = \mathcal{P} \backslash \{0\}$. Assume that $S : \overline{U} \to E$ is a completely continuous mapping and $T : \Omega \to E$ is an expansive one with constant $h > 1$.

Now, we present a multiple fixed point theorem which will be used to ensure the existence of at least two nonnegative solutions to BVP (1.6). The proof of this theorem is based upon a recent fixed point index developed in [7].

Theorem 2.6 [17, Theorem 2.8]. *Let U_1, U_2, and U_3 three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T : \Omega \to E$ is an expansive mapping with constant $h > 1$, $S : \overline{U}_3 \to E$ is a completely continuous mapping, and $S(\overline{U}_3) \subset (I - T)(\Omega)$. Suppose that $(U_2 \backslash \overline{U}_1) \cap \Omega \neq \emptyset$, $(U_3 \backslash \overline{U}_2) \cap \Omega \neq \emptyset$, and there exists $u_0 \in \mathcal{P}^*$ such that the following conditions hold:*

(i) *$S[u] \neq (I - T)[u - \lambda u_0]$ for all $\lambda > 0$ and $u \in \partial U_1 \cap (\Omega + \lambda u_0)$,*
(ii) *there exists $\varepsilon > 0$ small enough such that $S[u] \neq (I - T)[\lambda u]$ for all $\lambda \geq 1 + \varepsilon$, $u \in \partial U_2$, and $\lambda u \in \Omega$,*
(iii) *$S[u] \neq (I - T)[u - \lambda u_0]$ for all $\lambda > 0$ and $u \in \partial U_3 \cap (\Omega + \lambda u_0)$.*

Then, $T + S$ has at least two nonzero fixed points $u_1, u_2 \in \mathcal{P}$ such that

$$u_1 \in \partial U_2 \cap \Omega \text{ and } u_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$$

or

$$u_1 \in (U_2 \setminus U_1) \cap \Omega \text{ and } u_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega.$$

3 Main Results

We will start with the following useful lemmas, which give an integral representation of a solution of BVP (1.6).

Lemma 3.1 *If $x \in E$ is a solution of the problem*

$$\begin{aligned} x'(t) &= f(t, x(t)), \quad t \in [0, T], \\ \Delta x(t_k) &= I_k(x(t_k)), \quad k = 1, 2, 3, \dots, p, \end{aligned} \tag{3.1}$$

then it satisfies the integral equation

$$\begin{aligned} x(t) = e^{-Dt} x(0) &+ \int_0^t e^{-D(t-s)} \left(Dx(s) + f(s, x(s))\right) ds \\ &+ \sum_{0 < t_k < t} e^{-D(t-t_k)} I_k(x(t_k)), \quad t \in [0, T], \end{aligned} \tag{3.2}$$

and the conversely.

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Proof Firstly, we will note that the solution of the equation

$$x'(t) = -Dx(t) + Dx(t) + f(t, x(t)), \quad t \in [0, t_1],$$

is given by

$$x(t) = e^{-Dt}x(0) + \int_0^t e^{-D(t-s)}(Dx(s) + f(s, x(s)))ds, \quad t \in [0, t_1].$$

In particular, we have

$$x(t_1) = e^{-Dt_1}x(0) + \int_0^{t_1} e^{-D(t_1-s)}(Dx(s) + f(s, x(s)))ds.$$

Now, we consider the problem

$$\begin{aligned} x'(t) &= -Dx(t) + Dx(t) + f(t, x(t)), \quad t \in (t_1, t_2], \\ x(t_1^+) &= e^{-Dt_1}x(0) + \int_0^{t_1} e^{-D(t_1-s)}(Dx(s) + f(s, x(s)))ds + I_1(x(t_1)). \end{aligned}$$

For its solution, we have the representation

$$\begin{aligned} x(t) &= e^{-D(t-t_1)}x(t_1^+) + \int_{t_1}^t e^{-D(t-s)}(Dx(s) + f(s, x(s)))ds \\ &= e^{-D(t-t_1)}\left(e^{-Dt_1}x(0) + \int_0^{t_1} e^{-D(t_1-s)}(Dx(s) + f(s, x(s)))ds\right) \\ &\quad + \int_{t_1}^t e^{-D(t-s)}(Dx(s) + f(s, x(s)))ds + e^{-D(t-t_1)}I_1(x(t_1)) \\ &= e^{-Dt}x(0) + \int_0^{t_1} e^{-D(t-s)}(Dx(s) + f(s, x(s)))ds \\ &\quad + \int_{t_1}^t e^{-D(t-s)}(Dx(s) + f(s, x(s)))ds + e^{-D(t-t_1)}I_1(x(t_1)) \\ &= e^{-Dt}x(0) + \int_0^t e^{-D(t-s)}(Dx(s) + f(s, x(s)))ds \\ &\quad + e^{-D(t-t_1)}I_1(x(t_1)), \quad t \in (t_1, t_2]. \end{aligned}$$

Assume that the solution of the problem

$$\begin{aligned} x'(t) &= -Dx(t) + Dx(t) + f(t, x(t)), \quad t \in (t_{k-1}, t_k], \\ x(t_{k-1}^+) &= e^{-Dt_{k-1}}x(0) + \int_0^{t_{k-1}} e^{-D(t_{k-1}-s)}(Dx(s) + f(s, x(s)))ds \\ &\quad + \sum_{0<t_l<t_{k-1}} e^{-D(t-t_l)}I_l(x(t_l)) + I_{k-1}(x(t_{k-1})) \end{aligned}$$

for some $k \in \{1, \dots, p-1\}$, is given by

$$x(t) = e^{-Dt}x(0) + \int_0^t e^{-D(t-s)}(Dx(s) + f(s, x(s)))ds$$
$$+ \sum_{0<t_l<t_k} e^{-D(t-t_l)} I_l(x(t_l)), \quad t \in (t_{k-1}, t_k].$$

Then

$$x(t_k) = e^{-Dt_k}x(0) + \int_0^{t_k} e^{-D(t_k-s)}(Dx(s) + f(s, x(s)))ds$$
$$+ \sum_{0<t_l<t_k} e^{-D(t_k-t_l)} I_l(x(t_l)).$$

Now, we consider the problem

$$x'(t) = -Dx(t) + Dx(t) + f(t, x(t)), \quad t \in (t_k, t_{k+1}],$$
$$x(t_k^+) = e^{-Dt_k}x(0) + \int_0^{t_k} e^{-D(t_k-s)}(Dx(s) + f(s, x(s)))ds$$
$$+ \sum_{0<t_l<t_k} e^{-D(t_k-t_l)} I_l(x(t_l)) + I_k(x(t_k)).$$

For its solution, we have the following representation

$$\begin{aligned}
x(t) &= e^{-D(t-t_k)}x(t_k^+) + \int_{t_k}^t e^{-D(t-s)}(Dx(s) + f(s, x(s)))ds \\
&\quad + e^{-D(t-t_k)} I_k(x(t_k)) \\
&= e^{-D(t-t_k)}\Big(e^{-Dt_k}x(0) + \int_0^{t_k} e^{-D(t_k-s)}(Dx(s) + f(s, x(s)))ds \\
&\quad + \sum_{0<t_l<t_k} e^{-D(t_k-t_l)} I_l(x(t_l))\Big) \\
&\quad + \int_{t_k}^t e^{-D(t-s)}(Dx(s) + f(s, x(s)))ds + e^{-D(t-t_k)} I_k(x(t_k)) \\
&= e^{-Dt}x(0) + \int_0^{t_k} e^{-D(t-s)}(Dx(s) + f(s, x(s)))ds \\
&\quad + \int_{t_k}^t e^{-D(t-s)}(Dx(s) + f(s, x(s)))ds + \sum_{0<t_l<t_{k+1}} e^{-D(t-t_l)} I_l(x(t_l)) \\
&= e^{-Dt}x(0) + \int_0^t e^{-D(t-s)}(Dx(s) + f(s, x(s)))ds \\
&\quad + \sum_{0<t_l<t_{k+1}} e^{-D(t-t_l)} I_l(x(t_l)), \quad t \in (t_k, t_{k+1}].
\end{aligned}$$

Thus, the representation (3.2) holds. Now, assume that $x \in E$ satisfies (3.2). Fix $k \in \{1, \ldots, p\}$ arbitrarily and let $t \in (t_k, t_{k+1}]$. Then

$$\begin{aligned} x'(t) &= -D\Big(e^{-Dt}x(0) + \int_0^t e^{-D(t-s)}(Dx(s) + f(s, x(s)))ds \\ &\quad + \sum_{0<t_l<t_{k+1}} e^{-D(t-t_l)} I_l(x(t_l))\Big) + Dx(t) + f(t, x(t)) \\ &= -Dx(t) + Dx(t) + f(t, x(t)) \\ &= f(t, x(t)) \end{aligned}$$

and

$$\begin{aligned} x(t_k) &= e^{-Dt_k}x(0) + \int_0^{t_k} e^{-D(t_k-s)}(Dx(s) + f(s, x(s)))ds \\ &\quad + \sum_{0<t_l<t_k} e^{-D(t_k-t_l)} I_l(x(t_l)), \\ x(t_k^+) &= e^{-Dt_k}x(0) + \int_0^{t_k} e^{-D(t_k-s)}(Dx(s) + f(s, x(s)))ds \\ &\quad + \sum_{0<t_l<t_{k+1}} e^{-D(t_k-t_l)} I_l(x(t_l)). \end{aligned}$$

Hence,

$$\Delta x(t_k) = I_k(x(t_k)),$$

i.e., x satisfies (3.1). This completes the proof. □

Lemma 3.2 *If $x \in E$ is a solution to the BVP (1.6), then x satisfies the integral equation*

$$\begin{aligned} x(t) &= -e^{-Dt}(A + Be^{-DT})^{-1}B\int_0^T e^{-D(T-s)}\,(Dx(s) + f(s, x(s)))\,ds \\ &\quad - \sum_{0<t_k<T} e^{-Dt}e^{-D(T-t_k)}(A + Be^{-DT})^{-1}BI_k(x(t_k)) \\ &\quad + e^{-Dt}(A + Be^{-DT})^{-1}\left(\int_0^T g(s, x(s))ds - \int_0^T h(s)x(s)ds\right) \\ &\quad + \int_0^t e^{-D(t-s)}\,(Dx(s) + f(s, x(s)))\,ds \\ &\quad + \sum_{0<t_k<t} e^{-D(t-t_k)} I_k(x(t_k)), \quad t \in [0, T], \end{aligned} \tag{3.3}$$

and the conversely.

Proof Suppose that x is a solution of the BVP (1.6). We rewrite the first equation of (1.6) in the form

$$x'(t) = Dx(t) - Dx(t) + f(t, x(t))$$

and using Lemma 3.1, we have that x satisfies (3.2). Hence,

$$\begin{aligned} x(T) &= e^{-DT}x(0) + \int_0^T e^{-D(T-s)}\left(Dx(s) + f(s, x(s))\right) ds \\ &\quad + \sum_{0<t_k<T} e^{-D(T-t_k)} I_k(x(t_k)), \quad t \in [0, T] \end{aligned}$$

and

$$\begin{aligned} Bx(T) &= Be^{-DT}x(0) + B\int_0^T e^{-D(T-s)}\left(Dx(s) + f(s, x(s))\right) ds \\ &\quad + \sum_{0<t_k<T} e^{-D(T-t_k)} BI_k(x(t_k)), \quad t \in [0, T]. \end{aligned}$$

From this, the boundary condition in (1.6) becomes

$$\begin{aligned} &\int_0^T g(s, x(s))ds - \int_0^T h(s)x(s)ds \\ &\quad = Ax(0) + Bx(T) \\ &\quad = \left(A + Be^{-DT}\right)x(0) + B\int_0^T e^{-D(T-s)}\left(Dx(s) + f(s, x(s))\right) ds \\ &\quad + \sum_{0<t_k<T} e^{-D(T-t_k)} BI_k(x(t_k)), \end{aligned}$$

which yields

$$\begin{aligned} (A + Be^{-DT})x(0) &= -B\int_0^T e^{-D(T-s)}\left(Dx(s) + f(s, x(s))\right) ds \\ &\quad - \sum_{0<t_k<T} e^{-D(T-t_k)} BI_k(x(t_k)) \\ &\quad + \int_0^T g(s, x(s))ds - \int_0^T h(s)x(s)ds \end{aligned}$$

and consequently,

$$\begin{aligned} x(0) &= -(A + Be^{-DT})^{-1}B\int_0^T e^{-D(T-s)}\left(Dx(s) + f(s, x(s))\right) ds \\ &\quad - \sum_{0<t_k<T} e^{-D(T-t_k)}(A + Be^{-DT})^{-1}BI_k(x(t_k)) \\ &\quad + (A + Be^{-DT})^{-1}\left(\int_0^T g(s, x(s))ds - \int_0^T h(s)x(s)ds\right) \end{aligned}$$

Now, substituting this value in (3.2), we get (3.3). On the other hand, if $x \in E$ is a solution to the integral equation (3.3), then by direct computations, we see that x will be also a solution of BVP (1.6). This completes the proof. □

Before moving further, for $x \in E$, we define the operator $S_1 \colon E \to \mathbb{R}^n$ as

$$\begin{aligned} S_1[x](t) := &-e^{-Dt}(A+Be^{-DT})^{-1}B\int_0^T e^{-D(T-s)}\left(Dx(s)+f(s,x(s))\right)ds \\ &-\sum_{0<t_k<T} e^{-Dt}e^{-D(T-t_k)}(A+Be^{-DT})^{-1}BI_k(x(t_k)) \\ &+e^{-Dt}(A+Be^{-DT})^{-1}\left(\int_0^T g(s,x(s))ds-\int_0^T h(s)x(s)ds\right) \\ &+\int_0^t e^{-D(t-s)}\left(Dx(s)+f(s,x(s))\right)ds \\ &+\sum_{0<t_k<t} e^{-D(t-t_k)}I_k(x(t_k))-x(t), \quad t\in[0,T]. \end{aligned} \tag{3.4}$$

***Remark* 3.3** If $x \in E$ satisfies the equation $S_1 x = 0$, then it is a solution to the BVP (1.6).

Lemma 3.4 *Suppose that the hypotheses* (H_1)–(H_5) *hold. Then for any* $x \in E$ *with* $\|x\| \le D$*, the following inequality holds.*

$$\begin{aligned} \|S_1[x]\| \le D &+ \left(\|(A+Be^{-DT})^{-1}B\|+1\right)\sum_{j=1}^{n}\left(D^2T+TD\left(1+\sqrt{n}D^{p_j}\right)\right) \\ &+\left(\|\left(A+Be^{-DT}\right)^{-1}\|+1\right)D\sum_{j=1}^{n}(1+\sqrt{n}D^{q_j}) \\ &+\|(A+Be^{-DT})^{-1}\|T\left(D\sum_{j=1}^{n}(1+\sqrt{n}D^{r_j})+nD^2\right) \\ =:&\, D_1, \end{aligned}$$

where S_1 *is defined in* (3.4).

Proof In view of (H_1), (H_2), and (H_4), we write

$$\begin{aligned} |f_j(t,x(t))| &\le D(1+\sqrt{n}D^{p_j}), \\ |I_j(x(t_j))| &\le D(1+\sqrt{n}D^{q_j}), \text{ and} \\ |g_j(t,x(t))| &\le D\left(1+\sqrt{n}D^{r_j}\right), \quad j\in\{1,\ldots,n\}, \quad t\in[0,T]. \end{aligned}$$

Then

$$\|S_1[x]\| = \Bigg\| -e^{-Dt}(A+Be^{-DT})^{-1}B\int_0^T e^{-D(T-s)}\left(Dx(s)+f(s,x(s))\right)ds$$

$$
\begin{aligned}
&- \sum_{0<t_k<T} e^{-Dt} e^{-D(T-t_k)} (A+Be^{-DT})^{-1} B I_k(x(t_k)) \\
&+ e^{-Dt}(A+Be^{-DT})^{-1} \left(\int_0^T g(s,x(s))ds - \int_0^T h(s)x(s)ds \right) \\
&+ \int_0^t e^{-D(t-s)} \left(Dx(s) + f(s,x(s)) \right) ds + \sum_{0<t_k<t} e^{-D(t-t_k)} I_k(x(t_k)) - x(t) \Bigg\| \\
\le{}& \|x\| + \Bigg\| - e^{-Dt}(A+Be^{-DT})^{-1} B \int_0^T e^{-D(T-s)} \left(Dx(s) + f(s,x(s)) \right) ds \\
&- \sum_{0<t_k<T} e^{-Dt} e^{-D(T-t_k)} (A+Be^{-DT})^{-1} B I_k(x(t_k)) \\
&+ e^{-Dt}(A+Be^{-DT})^{-1} \left(\int_0^T g(s,x(s))ds - \int_0^T h(s)x(s)ds \right) \\
&+ \int_0^t e^{-D(t-s)} \left(Dx(s) + f(s,x(s)) \right) ds + \sum_{0<t_k<t} e^{-D(t-t_k)} I_k(x(t_k)) \Bigg\| \\
\le{}& \|x\| + \left\| - e^{-Dt}(A+Be^{-DT})^{-1} B \int_0^T e^{-D(T-s)} \left(Dx(s) + f(s,x(s)) \right) ds \right\| \\
&+ \left\| \sum_{0<t_k<T} e^{-Dt} e^{-D(T-t_k)} (A+Be^{-DT})^{-1} B I_k(x(t_k)) \right\| \\
&+ \left\| e^{-Dt}(A+Be^{-DT})^{-1} \left(\int_0^T g(s,x(s))ds - \int_0^T h(s)x(s)ds \right) \right\| \\
&+ \left\| \int_0^t e^{-D(t-s)} \left(Dx(s) + f(s,x(s)) \right) ds \right\| + \left\| \sum_{0<t_k<t} e^{-D(t-t_k)} I_k(x(t_k)) \right\|
\end{aligned}
$$

and subsequently, we obtain $\|S_1[x]\| \le D_1$. This completes the proof. □

Now, we introduce a new hypothesis and an operator as follows.

(H_6) Suppose $C > 0$ is a constant such that $CD_1 < D$.

For $x \in E$, define the operator $S_2 \colon E \to \mathbb{R}^n$ as

$$S_2[x](t) := \frac{C}{T} \int_0^t S_1[x](s)ds, \quad t \in [0, T], \tag{3.5}$$

where S_1 is defined as (3.4).

Remark 3.5 If $x \in E$ satisfies the equation

$$S_2[x](t) = Q, \quad t \in [0, T], \tag{3.6}$$

for arbitrary constant Q, then x is a solution to the problem (1.6). Really, differentiating (3.6) with respect to t, we get

$$\frac{C}{T} S_1[x](t) = 0, \quad t \in [0, T],$$

whereupon $S_1[x](t) = 0$, $t \in [0, T]$.

Lemma 3.6 *Suppose that* (H_1)–(H_5) *hold. Let* $x \in E$ *be such that* $\|x\| \leq D$. *Then*

$$\|S_2[x]\| \leq CD_1. \tag{3.7}$$

Proof The assertion follows directly from Lemma 3.4. □

3.1 Existence of at Least One Solution

Let $\widetilde{X}$ be the set of all equicontinuous families in E and define

$$X := \{x \in \widetilde{X} : \|x\| \leq D\}.$$

Let also, $\varepsilon \in \left(0, \frac{1}{2}\right)$. For $x \in E$, we define two operators $T, S : E \to \mathbb{R}^n$ as follows,

$$T[x](t) := -\varepsilon x(t) \tag{3.8}$$

and

$$S[x](t) := (1+\varepsilon)[x](t) + \varepsilon S_2[x](t), \quad t \in [0, T], \tag{3.9}$$

where S_2 is defined in (3.5). Note that any fixed point of the operator $T + S$ is a solution to the BVP (1.6).

Lemma 3.7 *Suppose that* (H_1)–(H_5) *and* (H_6) *hold. Then for* $x \in X$, *the following inequalities hold.*

$$\|(I - S)[x]\| \leq D \quad \textit{and} \quad \|((1+\varepsilon)I - S)[x]\| < \epsilon D.$$

Proof Using definition of the operator S, we have

$$\begin{aligned}
\|(I - S)[x]\| &= \|-\varepsilon x - \varepsilon S_2[x]\| \\
&\leq \varepsilon\|x\| + \varepsilon\|S_2[x]\| \\
&\overset{(3.7)}{\leq} \varepsilon(D + CD_1) \\
&\leq D
\end{aligned}$$

and

$$\begin{aligned}
\|((1+\varepsilon)I - S)[x]\| &= \|\varepsilon S_2[x]\| \\
&= \varepsilon\|S_2[x]\| \\
&\overset{(3.7)}{\leq} \varepsilon CD_1 \\
&< \epsilon D.
\end{aligned}$$

This completes the proof. □

Now, we are in a position to state and prove our main result of this section which is as follows.

Theorem 3.8 *Suppose that* (H_1)*–*(H_5) *and* (H_6) *hold. Then the BVP* (1.6) *has at least one solution in* E.

Proof Note that the operators $S_1, S_2 : E \to \mathbb{R}^n$ defined in (3.4) and (3.5) respectively, are continuous operators. We consider two operators T and S defined in (3.8) and (3.9) respectively. In view of Lemma 3.7, it follows that the operator $I - S : X \to X$ and it is continuous. Since the continuous maps of equicontinuous families forms equicontinuous families, we conclude that the image set $(I - S)(X)$ resides in a compact subset of E. Suppose that there is an element $x \in \partial X$ and $\lambda \in \left(0, \frac{1}{\varepsilon}\right)$ satisfying

$$\lambda(I - S)[x] = x. \tag{3.10}$$

This, using definition of operator S, we write as

$$\frac{1}{\lambda}x = -\varepsilon x - \varepsilon S_2[x].$$

That is,

$$\left(\frac{1}{\lambda} + \varepsilon\right) x = ((1 + \varepsilon)I - S)[x].$$

Whereupon we have the following

$$\varepsilon D < \left(\frac{1}{\lambda} + \varepsilon\right) D = \left(\frac{1}{\lambda} + \varepsilon\right) \|x\| = \varepsilon \| S_2[x]\| = \|((1 + \varepsilon)I - S)[x]\|.$$

But from Lemma 3.7, we have $\|((1 + \varepsilon)I - S)[x]\| < \varepsilon D$. Hence, we arrived at a contradiction. Therefore, our assumption that (3.10) is not correct. From here, we get

$$\{u \in X : u = \lambda(I - S)[u], \quad \|u\| = D\} = \emptyset \quad \text{for any} \quad \lambda \in \left(0, \frac{1}{\epsilon}\right).$$

Thus, all conditions of Theorem 2.1 hold. Therefore, the operator $T + S$ has a fixed point in E which is a solution of BVP (1.6). Hence, BVP (1.6) has at least one solution. This completes the proof. □

3.2 Existence of at Least Two Nonnegative Solutions

Next, before proving the result concerning multiple nonnegative solutions of BVP (1.6), we introduce one more hypothesis as follows.

(H_7) Let $m > 0$ be large enough and r, L, D be positive constants that satisfy the following inequalities.

(i) $r < L < D$,

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(ii) $D > \left(\frac{2}{5m} + 1\right) L$,

(iii) $CD_1 < \frac{L}{5}$.

Let $\varepsilon > 0$. For $x \in E$, we define two operators $T_1, S_3 \colon E \to \mathbb{R}^n$ as follows,

$$T_1[x](t) := (1 + m\varepsilon)x(t) - \varepsilon\frac{L}{10} \tag{3.11}$$

and

$$S_3[x](t) := -\varepsilon S_2 x(t) - m\varepsilon x(t) - \varepsilon\frac{L}{10}, \quad t \in [0, T], \tag{3.12}$$

where S_2 is defined in (3.5). Note that both T_1 and S_3 are continuous operators on E. Further, any fixed point $x \in E$ of the operator $T_1 + S_3$ is solution to the BVP (1.6). Our main result in this section is as follows.

Theorem 3.9 *Suppose that* (H_1)*–*(H_5) *and* (H_7) *hold. Then BVP* (1.6) *has at least two nonnegative solutions in* E.

Proof Let $\mathcal{P} := \{x \in \widetilde{X} \colon x \geq 0\}$ be a cone in E. First we define some subsets of $\mathcal{P}$ which are used in the proof.

$$\begin{aligned}
U_1 &= \mathcal{P}_r := \{x \in \mathcal{P} \colon \|x\| < r\},\\
U_2 &= \mathcal{P}_L := \{x \in \mathcal{P} \colon \|x\| < L\},\\
U_3 &= \mathcal{P}_D := \{x \in \mathcal{P} \colon \|x\| < D\},\\
\Omega &= \overline{\mathcal{P}}_{R_2} := \left\{x \in \mathcal{P} : \|x\| \leq R_2, \quad R_2 = D + \frac{C}{m}D_1 + \frac{L}{5m}\right\}.
\end{aligned}$$

Now, we consider two operators T_1 and S_3 defined in (3.11) and (3.12) respectively, and employ Theorem 2.6. The proof will be given in the following steps.

Step 1. For $x, y \in \Omega$, from (3.11) we see that

$$\|T_1[x] - T_1[y]\| = (1 + m\varepsilon)\|x - y\|,$$

whereupon the operator $T_1 \colon \Omega \to \mathbb{R}^n$ is an expansive with a constant $1 + m\varepsilon > 1$.

Step 2. For $x \in \overline{U}_3$, from (3.12) we see that

$$\begin{aligned}
\|S_3[x]\| &\leq \varepsilon\|S_2[x]\| + m\varepsilon\|x\| + \varepsilon\frac{L}{10}\\
&\overset{(3.7)}{\leq} \varepsilon\left(CD_1 + mD + \frac{L}{10}\right).
\end{aligned}$$

This means that $S_3(\overline{U}_3)$ is uniformly bounded. Now, since $S_3 \colon \overline{U}_3 \to \mathbb{R}^n$ is continuous, it follows that $S_3(\overline{U}_3)$ is equicontinuous. Consequently, the operator $S_3 : \overline{U}_3 \to \mathbb{R}^n$ is completely continuous.

Step 3. For $x \in \overline{U}_3$, set

$$y = x + \frac{1}{m} S_2[x] + \frac{L}{5m}. \tag{3.13}$$

Note that $S_2[x](t) + \frac{L}{5} \geq 0$ on $[0, T]$. Then, we have $y \geq 0$ on $[0, T]$ and

$$\begin{aligned} \|y\| &\leq \|x\| + \frac{1}{m}\|S_2[x]\| + \frac{L}{5m} \\ &\overset{(3.7)}{\leq} D + \frac{C}{m} D_1 + \frac{L}{5m} \\ &= R_2. \end{aligned}$$

Thus, $y \in \Omega$ and from (3.13), we write

$$-\varepsilon m y = -\varepsilon m x - \varepsilon S_2[x] - \varepsilon \frac{L}{10} - \varepsilon \frac{L}{10}$$

which, upon using (3.11) and (3.12), yields

$$(I - T_1)[y] = S_3[x].$$

Thus, $S_3(\overline{U}_3) \subset (I - T_1)(\Omega)$.

Step 4. Assume that for any $u_0 \in \mathcal{P}^*$ there exist $\lambda > 0$ and $x \in \partial \overline{U}_1 \cap (\Omega + \lambda u_0)$ or $x \in \partial \overline{U}_3 \cap (\Omega + \lambda u_0)$ such that

$$S_3[x] = (I - T_1)[x - \lambda u_0].$$

Then using (3.11) and (3.12), we write this as

$$-\varepsilon S_2[x] - m\varepsilon x - \varepsilon \frac{L}{10} = -m\varepsilon(x - \lambda u_0) + \varepsilon \frac{L}{10},$$

which yields

$$-S_2[x] = \lambda m u_0 + \frac{L}{5}.$$

Therefore

$$\|S_2[x]\| = \left\| \lambda m u_0 + \frac{L}{5} \right\| > \frac{L}{5},$$

which is a contradiction. Hence,

$$S_3[x] \neq (I - T_1)[x - \lambda u_0]$$

for all $\lambda > 0$ and $x \in \partial \overline{U}_1 \cap (\Omega + \lambda u_0)$ or $x \in \partial \overline{U}_3 \cap (\Omega + \lambda u_0)$.

Step 5. Let $\varepsilon_1 = \frac{2}{5m}$. Assume that there exist $\lambda_1 \geq \varepsilon_1 + 1$, $x_1 \in \partial U_2$, and $\lambda_1 x_1 \in \Omega$ such that

$$S_3[x_1] = (I - T_1)[\lambda_1 x_1]. \tag{3.14}$$

Since $x_1 \in \partial U_2$ and $\lambda_1 x_1 \in \Omega$, it follows that

$$\left(\frac{2}{5m}+1\right)L \le \lambda_1 L = \lambda_1\|x_1\| \le R_2.$$

Moreover, using (3.11) and (3.12), we have

$$-\varepsilon S_2[x_1] - m\varepsilon x_1 - \varepsilon\frac{L}{10} = -\lambda_1 m\varepsilon x_1 + \varepsilon\frac{L}{10}.$$

That is,

$$S_2[x_1] + \frac{L}{5} = (\lambda_1 - 1)mx_1.$$

From here,

$$2\frac{L}{5} > \left\|S_2x_1 + \frac{L}{5}\right\| = (\lambda_1 - 1)m\|x_1\| = (\lambda_1 - 1)mL$$

and

$$\frac{2}{5m} + 1 > \lambda_1,$$

which is a contradiction. Hence, there exists $\varepsilon > 0$ small enough such that $S_3[x_1] \neq (I - T_1)[\lambda_1 x_1]$ for all $\lambda_1 \ge \varepsilon + 1$, $x_1 \in \partial U_2$, and $\lambda_1 x_1 \in \Omega$.

From above steps, we see that all conditions of Theorem 2.6 hold. Hence, the BVP (1.6) has at least two solutions, say $u_1, u_2 \in E$ such that

$$\|u_1\| = L < \|u_2\| < D$$

or

$$r < \|u_1\| < L < \|u_2\| < D.$$

This completes the proof. □

4 An Illustrative Example

In this section, we shall give an example that show the utility of our main theorems 3.8 and 3.9. Consider following the boundary value problem.

$$\begin{aligned}
&x_1'(t) = \frac{(x_2(t))^2}{1+(x_2(t))^4} \text{ and } x_2'(t) = \frac{(x_1(t))^3}{1+(x_1(t))^6}, \quad t \in [0,1],\ t \neq \frac{1}{3},\\
&\Delta x_1\left(\frac{1}{3}\right) = \frac{\left(x_2\left(\frac{1}{3}\right)\right)^4}{1+\left(x_2\left(\frac{1}{3}\right)\right)^8} \text{ and } \Delta x_2\left(\frac{1}{3}\right) = \frac{\left(x_1\left(\frac{1}{3}\right)\right)^6}{1+\left(x_1\left(\frac{1}{3}\right)\right)^{10}},\\
&x_1(0) - x_2(0) + \int_0^1 \frac{s^2}{1+s^4}x_1(s)ds + x_1(1) = \int_0^1 \frac{(x_1(s))^2}{1+(x_1(s))^4}ds,\\
&-x_1(0) - x_2(0) + \int_0^1 \frac{s^2}{1+s^4}x_2(s)ds + x_2(1) = \int_0^1 \frac{(x_2(s))^3}{1+(x_2(s))^6}ds. \quad (4.1)
\end{aligned}$$

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Comparing BVP (4.1) with BVP (1.6), we find that $T = 1$, $p = 1$, $t_1 = \frac{1}{3}$,

$$
\begin{aligned}
&f_1(t, v) = \frac{v_2^4}{1 + v_2^4}, \quad f_2(t, v) = \frac{v_1^3}{1 + v_1^6}, \quad I_{11}(v) = \frac{v_2^3}{1 + v_2^8}, \\
&I_{12}(v) = \frac{v_1^6}{1 + v_1^{10}}, \quad h(t) = \frac{t^2}{1 + t^4}, \\
&g_1(t, v) = \frac{v_1^2}{1 + v_1^4}, \quad g_2(t, v) = \frac{v_2^3}{1 + v_2^6}, \quad t \in [0, 1], \quad v = (v_1, v_2), \\
&A = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\end{aligned}
$$

and $p_1 = 4$, $p_2 = 3$, $q_1 = 4$, $q_2 = 6$, $a_{11}(t) = a_{12}(t) = 0$, $a_{21}(t) = a_{22}(t) = 1$, $a_{111} = 0$, $a_{211} = 1$, $a_{112} = 0$, $a_{212} = 1$, $D = 1$. Next, we see that

$$
\begin{aligned}
(A + e^{-1}B) &= (A + e^{-1}B)B \\
&= \begin{pmatrix} 1 + e^{-1} & -1 \\ -1 & -1 + e^{-1} \end{pmatrix} \text{ with } \det(A + e^{-1}B) = -2 + e^{-2},
\end{aligned}
$$

and

$$
(A + e^{-1}B)^{-1}B = (A + e^{-1}B)^{-1} = \frac{1}{e^{-2} - 2} \begin{pmatrix} -1 + e^{-1} & 1 \\ 1 & 1 + e^{-1} \end{pmatrix},
$$

with

$$
\|(A + e^{-1}B)^{-1}\| = \frac{e^{-2} - 2}{e^{-2} - 2} = 1.
$$

Further,

$$
\begin{aligned}
D_1 &= 1 + 2(1 + 1)(1 + 1 + \sqrt{2}) + 2(1 + 1)(1 + \sqrt{2}) + (2(1 + \sqrt{2}) + 2) \\
&= 17 + 10\sqrt{2}.
\end{aligned}
$$

Take $C = \frac{1}{10^{10000}}$. Then we get $CD_1 < D$. Thus, (H_1)–(H_6) hold. Hence, employing Theorem 3.8, we conclude that the considered BVP (4.1) has at least one solution. Let now $L = \frac{1}{6}$, $r = \frac{1}{8}$, $m = 10^{50}$. With this data, it follows that

$$
r < L < D, \quad 1 = D > \left(\frac{2}{(5)(10^{50})} + 1 \right) \frac{1}{6} = \left(\frac{2}{5m} + 1 \right) L,
$$

and $CD_1 < \frac{L}{5}$. Thus, (H_7) also holds. Then, by employing Theorem 3.9, we conclude that the BVP (4.1) has at least two nonnegative solutions.

5 Concluding Remarks

This paper explores the existence of at least one solution and at least two nonnegative solutions for the differential equations with impulses and two-point integral boundary conditions by employing recent fixed point theorems for the sum of two operators on Banach spaces. The results of this paper are essentially new in the sense that they are considered in the context of impulsive conditions and two-point integral boundary conditions. The boundary condition taken in this work is more general and includes various others as special cases. The results obtained in this paper can be further investigated for higher-order differential equations with impulses. We also believe that other qualitative properties like data dependence, stability, controllability, and oscillations can be studied in the forthcoming papers.

Acknowledgements The authors are thankful to the referees for their constructive comments and valuable suggestions towards improvement of the paper. The research work of Sanket Tikare is a part of the seed grant MRP (Ref. No. 886/13122021) of Ramniranjan Jhunjhunwala College, Mumbai, India.

Author Contributions All authors contributed equally and significantly in preparing and writing this article. All authors read and approved the final manuscript.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare that there is no competing interests regarding the publication of this article.

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