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Exact local fractional differential equations

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ABSTRACT

Here the theory of local fractional differential equations is extended to a more general class of equations akin to usual exact differential equations. In the process, a new notion has been introduced which is termed as α-exact local fractional differential equation. The theory of such equations parallels that for the first order ordinary differential equations. A criterion to check the α-exactness emerges naturally and also a method to find general solutions of such equations. This development completes the basic theory of the local fractional differential equations. The solved examples demonstrate how complex functions arise as solutions which will be useful in understanding the processes taking place on fractals.

1. Introduction

Ordinary differential equations played an important role in Mathematics as well as in several other fields where they found applications. During the development of calculus, the idea of derivatives of non-integer orders [1,2] was also proposed and, soon after the confusion surrounding its definition was cleared, the differential equations of fractional order [3,4] were also considered. These equations too have been applied in modelling various physical phenomena involving a long term memory [5–8]. There are multitude of definitions of the derivatives of fractional order which makes it necessary to study fractional differential equations using these different definitions and understand which situations can be better described by them. We had introduced a version of the fractional derivative termed as *local fractional derivative (LFD)* [9] which also led to the introduction of the *local fractional differential equations (LFDE)* involving these LFDs.

However, since the LFD has some properties which are quite different as compared to the other definitions, the equations involving them turn out to have a very different meaning and need to be interpreted very carefully. This makes the development little slow. As a result, though the first simple case was introduced in [10], the next development happened only in [11]. In [10], a generalisation of the simplest first order differential equation in which the unknown function depends only on the independent variable was carried out. The generalisation consisted of replacing the first order derivative with the LFD of order between 0 and 1. Interestingly this led to a new kind of equations in which one could incorporate phenomena on fractals [12] and solve the equation meaningfully. In the same work, the equation was applied to diffusion taking place in fractal time. The next step was to extend the theory of LFDEs to cases in which the unknown function depends on the dependent variable too. In [11], it was extended to separable local fractional differential equations. The present work takes the next important step in this program and develops the theory for exact LFDEs and thus, in a sense, completes the formal development of generalisation of the first order differential equations to the ones involving the LFD.

The organization of the paper is as follows. The next section gives a quick introduction to the previous developments which helps in defining the symbols and also to collect important facts needed later in the work. In Section 3 I introduce the concept through some examples and in Section 4 the theory is developed. A plausible physical application is considered in Section 5. Then, in the last section, there are some concluding remarks.

2. Local fractional derivative

In [9], the following definition of LFD was introduced:

$$\frac{df(x)}{dx^q} = \lim_{x' \to x} D_x^q(f(x') - f(x)) \quad (0 < q < 1) \tag{1}$$

where

$$D_x^q f(x') = \frac{1}{\Gamma(1-q)} \frac{d}{dx'} \int_x^{x'} f(t)(x'-t)^{-q} dt, \tag{2}$$

is the Riemann-Liouville fractional derivative of order q. Notice that the notation for the LFD has been changed. The order q does not appear in the numerator as there is only a first order difference of f. This notation will turn out to be convenient later in the paper when we generalise the notion of exactness. The reader is requested to refer to [13] for more insights about this definition and some worked out examples. Important points to remember

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are in order. Firstly, the LFD of a differentiable function is zero so it makes sense only to apply it to nondifferentiable functions. Secondly, the order of the derivative also has to be chosen equal to a critical order which is equal to the local Hölder exponent of the function at which the LFD is being evaluated. This definition was applied to a everywhere continuous but nowhere differentiable function and shown that the LFD is zero for orders less than the critical order and does not exist for orders greater.

It is difficult to establish what happens exactly at the critical order owing to the intrinsic oscillations any such function has [14]. Here it is assumed that one would be able to get around this problem either by taking some average behaviour or by somehow incorporating a way to take care of the oscillations into the definition itself. At the moment, the theory is developed symbolically assuming that there is a way to assign a finite value to the LFD at the critical order. It would be of interest to explore what kind of mathematical structure unfolds and also the possible applications this developments will have.

With this in mind, it is interesting to note some of the properties the LFD possesses which would be crucial to the developments ahead. The LFD admits a local fractional Taylor expansion [9]

$$f(x') = f(x) + \frac{1}{\Gamma(q+1)}\frac{df(x)}{dx^q}(x'-x)^q + R_q(x',x) \tag{3}$$

where $R_q(x',x)$ is a remainder term. It also satisfies the usual Leibnitz rule and the chain rule [15–17].

The next step is to consider differential equations involving the LFD and the simplest such equation is

$$\frac{dy}{dx^\alpha} = f(x) \tag{4}$$

where $0 < \alpha < 1$. This equation was introduced in [10]. We write the solution of this equation using a generalisation of Riemann sum:

$$y(x) = \lim_{N\to\infty}\sum_{i=0}^{N-1}\frac{(x_{i+1}-x_i)^\alpha}{\Gamma(\alpha+1)}f(x^*) \tag{5}$$

where $x_i \le x^* \le x_{i+1}$. The factor $(x_{i+1}-x_i)^\alpha/\Gamma(\alpha+1)$ has origins in the local fractional Taylor expansion (Eq. (3)). Now, a close inspection of the solution in (5) tells us that the solution can not exist if $f(x)$ is a continuous function [13]. One possibility when the solution exists is when the function $f(x)$ has a fractal support. It can be argued that if the dimension of the fractal support is equal to the order of the derivative then a finite solution exist. In particular, if $f(x) = 1_C(x)$, the indicator function of the fractal set C, then the solution is given by

$$y(x) = \lim_{N\to\infty}\sum_{i=0}^{N-1}\frac{(x_{i+1}-x_i)^\alpha}{\Gamma(\alpha+1)}F_C^i \equiv \frac{P_C(x)}{\Gamma(\alpha+1)} \equiv P_C^\alpha(x), \tag{6}$$

where F_C^i is a flag function which takes value 1 if the interval $[x_i, x_{i+1}]$ contains a point of the set C and 0 otherwise. Here, $P_C(x)$ is a Lebesgue-Cantor (staircase) function which is constant almost everywhere and rises only at points of the Cantor set C. This approach has been applied to Continuum Mechanics of disordered media [18–22].

Of course, more generally, one can choose $f(x)$ to be a product of any function and the indicator function. In this case too, the solution can be seen to exist. The next step is to consider f to depend on the dependent variable too. In [11], progress in this direction was made by considering separable local fractional differential equations. Since the LFD follows the chain rule, theory for this class of equations paralleled that for the ordinary differential equations. Moreover, it was not necessary to put any condition on the part depending on the dependent variable as was the case for that depending on the independent variable (that it has to have fractal support) for the solution to exist.

Now the new challenge is to extend the theory further, that is, to explore if a notion parallel to exactness of the ordinary differential equation can be meaningfully evolved.

3. Exact local fractional differential equations: understanding through examples

In this section, we show that the notion of exactness of the ordinary differential equations can be generalised and the theory can be extended to apply even to the local fractional differential equations. It parallels the theory of exact differential equations for the ordinary differential equations but the functions involved have to be chosen carefully. I demonstrate the point through examples. For this purpose, first the LFDE is written as follows:

$$M(x,y)dx^\alpha + N(x,y)dy = 0. \tag{7}$$

Clearly, this is equivalent to the LFDE

$$\frac{dy}{dx^\alpha} = -\frac{M(x,y)}{N(x,y)}. \tag{8}$$

If $N(x,y) = 1$ and $M(x,y) = f(x)$ then this equation is same as Eq. (4), so we know that $f(x)$ and hence $M(x,y)$ should have a fractal support in x. Further, when $M(x,y)$ is a product of two functions $f(x)$ and $g(y)$, we know how to solve it under the same condition on $f(x)$. But for more general $M(x,y)$ and nonseparable $N(x,y)$, it is not clear under which conditions, if any, a nontrivial solution would exist.

A useful hint is obtained from the following observation. In order to write an exact differential on the LHS of Eq. (7), we would have to write M as partial LFD of some f w. r. t. x and N as partial derivative of the same f w. r. t. y. Therefore to check for the exactness we would need to take the partial derivative of M w. r. t. y and a partial LFD of N w. r. t. x. Therefore, we see that, the N should be such that it would make sense after taking its LFD. One example of such a function is $P_C^\alpha(x)$.

So keeping this in mind let's consider couple of examples. The development of a proper theory is postponed to the next section. Our first example is the following simple looking equation but involves a major step forward.

$$1_C(x)y dx^\alpha + P_C^\alpha(x)dy = 0. \tag{9}$$

The function $P_C^\alpha(x)$ which was the solution of the simplest LFDE we considered in the last section now appears in our differential equation. Written out in a normal form it looks like

$$\frac{dy}{dx^\alpha} = -\frac{1_C(x)y}{P_C^\alpha(x)} \tag{10}$$

It is unclear how one could solve this equation. We'll develop the method in the next section but for now it can be checked that

$$y = \frac{c}{P_C^\alpha(x)} \tag{11}$$

solves the above equation.

Now let us consider another example of an equation which looks more complicated.

$$1_C(x)y dx^\alpha + ((P_C^\alpha(x))^2 y - P_C^\alpha(x))dy = 0 \tag{12}$$

In the next sections, we'll show that the solution of this equation is given by

$$\frac{y^2}{2} - \frac{y}{P_C^\alpha(x)} = c \tag{13}$$

These examples demonstrate that if one chooses the functions $M(x,y)$ and $N(x,y)$ correctly one can have meaningful solutions to the LFDE allowing one to expand the class of solutions.

4. Theory of exact local fractional differential equation

In this section, I develop the theory for exact local fractional differential equations. So consider the equation of the type

$$M(x, y)dx^{\alpha} + N(x, y)dy = 0. \tag{14}$$

and assume that there exist $f(x, y)$ such that

$$\frac{\partial f}{\partial x^{\alpha}} = M \quad \text{and} \quad \frac{\partial f}{\partial y} = N \tag{15}$$

Then Eq. (14) can be written as

$$\frac{\partial f}{\partial x^{\alpha}}dx^{\alpha} + \frac{\partial f}{\partial y}dy = 0 \quad \text{or} \quad df = 0. \tag{16}$$

Therefore the general solution is given by

$$f(x, y) = c.$$

We can call the differential equation as α-exact in order to show the dependence on α. Now, in order to get the condition for α-exactness, it is assumed that $f(x, y)$ is a smooth function of y and locally fractionally differentiable function of order α of x so that the mixed derivatives of f are equal, that is,

$$\frac{\partial^2 f}{\partial y \partial x^{\alpha}} = \frac{\partial^2 f}{\partial x^{\alpha} \partial y}. \tag{17}$$

This implies that the condition to check for α-exactness becomes

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x^{\alpha}}. \tag{18}$$

Now to find f, we integrate the first of Eq. (15) to obtain

$$f = \int M d^{\alpha}x + g(y). \tag{19}$$

To find the unknown function $g(y)$, we differentiate this f w. r. t. y and equate it to N to obtain

$$g'(y) = N - \frac{\partial}{\partial y}\int M d^{\alpha}x, \tag{20}$$

and hence

$$g(y) = \int \left(N - \frac{\partial}{\partial y}\int M d^{\alpha}x\right)dy. \tag{21}$$

In essence, it is shown that if the differential equation is α-exact then the condition (15) is true. The converse can also to be seen to be true. Moreover, it is shown that, in this case, its general solution is given by $f(x, y) = c$.

The next question deals with what happens if the LFDE is not exact. Can we find an integrating factor $\mu(x, y)$, and under which conditions, that we can multiply to make the LFDE exact? In order to answer this question, we multiply the differential equation by μ. Then we have

$$\mu M(x, y)dx^{\alpha} + \mu N(x, y)dy = 0 \tag{22}$$

and

$$\frac{\partial f}{\partial x^{\alpha}} = \mu M \quad \text{and} \quad \frac{\partial f}{\partial y} = \mu N \tag{23}$$

Therefore the condition for α-exactness becomes

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x^{\alpha}} \tag{24}$$

which, on expansion, becomes

$$\mu\frac{\partial M}{\partial y} + M\frac{\partial \mu}{\partial y} = \mu\frac{\partial N}{\partial x^{\alpha}} + N\frac{\partial \mu}{\partial x^{\alpha}}. \tag{25}$$

On rearranging, we get the following partial differential equation satisfied by μ

$$\frac{1}{\mu}\left(N\frac{\partial \mu}{\partial x^{\alpha}} - M\frac{\partial \mu}{\partial y}\right) = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x^{\alpha}}. \tag{26}$$

Any particular solution of this equation will give us the necessary integrating factor.

Now let us consider the first example, Eq. (9), of the last section. It can be checked that this equation is α-exact, that is,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x^{\alpha}} = 1_C(x). \tag{27}$$

Therefore the solution can be found using Eqs. (19) and (21). Therefore we have

$$f = \int M d^{\alpha}x + g(y) = P_C^{\alpha}(x)y + g(y) \tag{28}$$

where

$$g(y) = \int \left(N - \frac{\partial}{\partial y}\int M d^{\alpha}x\right)dy$$
$$= 0. \tag{29}$$

This leads to

$$f = P_C^{\alpha}(x)y = c \tag{30}$$

and therefore

$$y = \frac{c}{P_C^{\alpha}(x)}. \tag{31}$$

Now let us consider the second example, that is, Eq. (12). It can be checked that it is not α-exact so we assume that the integrating factor depends only on x and construct the quantity

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x^{\alpha}}}{N} = -2\frac{1_C(x)}{P_C^{\alpha}(x)}. \tag{32}$$

Therefore, solving the Eq. (26), the integrating factor is

$$\mu = e^{-2\int \frac{1_C(x)}{P_C^{\alpha}(x)} d^{\alpha}x} = e^{-2\ln P_C^{\alpha}(x)} = (P_C^{\alpha}(x))^{-2}.$$

With this the Eq. (12) becomes

$$(P_C^{\alpha}(x))^{-2} 1_C(x) y d^{\alpha}x + (P_C^{\alpha}(x))^{-2}((P_C^{\alpha}(x))^2 - P_C(x))dy = 0$$

and, using Eqs. (19) and (21), we get

$$f = \frac{y^2}{2} - \frac{y}{P_C^{\alpha}(x)}. \tag{33}$$

5. A plausible physical example

In order to illustrate the importance of this formalism, I consider an example motivated by a physical situation. Let us consider a variable mass system with no external force and moving with a non-zero velocity with respect to some frame of reference. The mass is being added to the system but the process of addition of mass takes place in fractal time. More specifically, we have

$$\frac{dm}{dt^{\alpha}} = 1_C(t). \tag{34}$$

This is possible when, for example, the addition of mass is a result of ejection from some self-organized critical process. So the masses are added intermittently with the time intervals distributed according to a power law. Therefore the total mass at time t is given by

$$m(t) = m_0 + P_C^{\alpha}(t) \equiv m_C^{\alpha}(t) \tag{35}$$

where m_0 is the mass of the system at $t = 0$. Now, clearly, ordinary calculus can not handle such a singular function and hence we can not use the Newton's law given in the usual form.

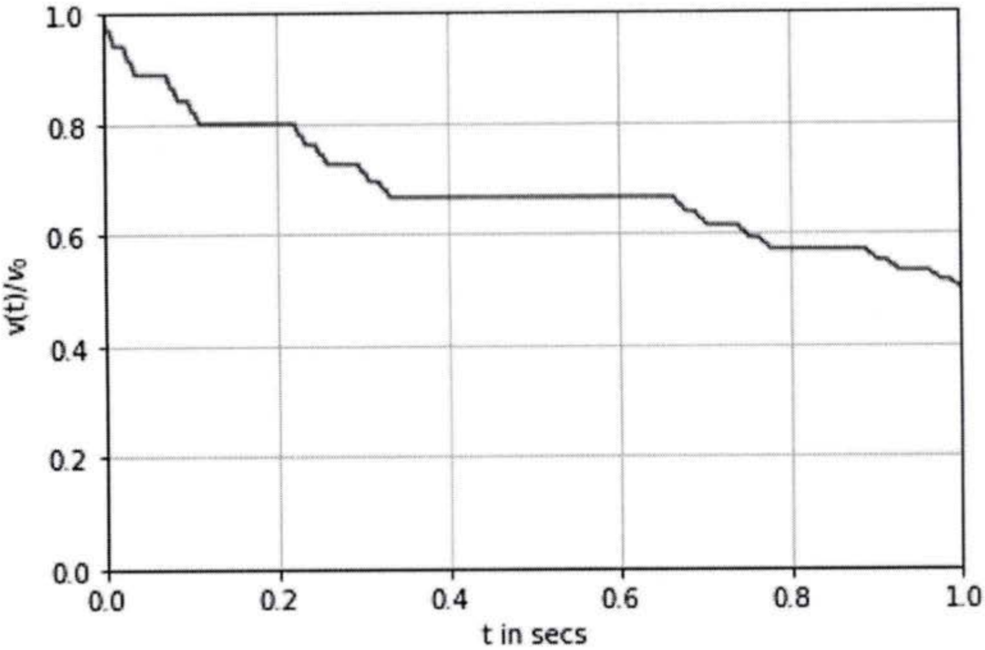

Fig. 1. The figure shows how the velocity (Eq. (39)) of an object would decrease when its mass increases intermittently on fractal time. Here the initial mass m_0 is chosen to be 1. The underlying fractal set is the middle-third Cantor set.

Let us assume that a "renormalized" Newton's second law is valid and we have for our system

$$\frac{dp}{dt^\alpha} = 0 \tag{36}$$

where $p = mv$ is the usual momentum. This leads us to

$$\frac{dm}{dt^\alpha} v + m\frac{dv}{dt^\alpha} = 0. \tag{37}$$

Using Eqs. (34) and (35), we get

$$1_C(t)v dt^\alpha + m_C^\alpha(t) dv = 0. \tag{38}$$

Now this is a differential equation which is the same as our first example. So we get the solution as

$$v(t) = \frac{v_0 m_0}{m_C^\alpha(t)}, \tag{39}$$

where v_0 is the velocity at $t = 0$. We see that the velocity decreases asymptotically as a power law in time. The velocity has been depicted in Fig. 1 for $m_0 = 1$ on the middle-third Cantor set.

6. Concluding remarks and future directions

In this article, we have extended the theory of local fractional differential equations to include "exact" equations. This completes the basic theory of the ordinary LFDEs. In the process, the concept of exact equations has been generalised to that of α-exact equations.

This was an important step further as it greatly expanded the types of equations one can consider and also the types of functions we can obtain as solutions. It was also a difficult step as it involved the realisation that the Lebesgue-Cantor staircase function, which was previously obtained as a solution of the simple LFDE, can be used in the equation itself to enlarge the class of equations and hence solutions.

This is also an important development from the point of view of applications as it paves a way to obtain complicated functions on fractal sets useful in many processes involving fractals, like diffusion or waves on fractals.

In this paper, we have considered the LFDE only for orders between 0 and 1. Here, it should be emphasized that the LFD does not satisfy the rule of composition and one can not consider the LFDE for higher orders. Moreover, the LFDE can be written only at the critical order. But this may not be a serious drawback as one can still consider a system of LFDEs thereby studying such equations in higher dimensions.

The importance of this work also lies in the new questions it has opened up. It suggests a way to generalise the divergence and curl operators leading to a generalisation of differential calculus. It also suggests a way to mathematically generalise the definition of the LFD. This stems from the observation that the way the LFDE has been written in this work has an asymmetric character. Only one of the differential has a fractional power. It would be interesting to wonder the meaning, provided there is an application, of making both differentials fractional.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Kiran M. Kolwankar: Conceptualization, Formal analysis, Writing – original draft, Funding acquisition.

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