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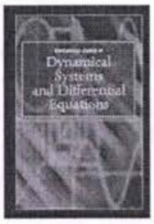

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Guest Editors: Prof. Jaqueline G. Mesquita, Prof. Martin Bohner, Prof. Carlos Lizama and Prof. Hideaki Matsunaga

Preface

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First-order nonlinear dynamic initial value problems

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Abstract: We prove three existence theorems for solutions of first-order dynamic initial value problems, including corresponding continuous and discrete cases. The main tools are fixed point theorems and dynamic inequalities. Two more results are given that discuss dependence of solutions on the initial conditions as well as convergence of sequences of solutions.

Keywords: time scales; dynamic equation; first-order nonlinear; existence; continuous dependence; fixed point theorems; dynamic inequalities.

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1 Introduction

For a time scale $\mathbb{T}$, we study the dynamic initial value problem

$$\begin{cases} x^{\Delta}(t) + \psi(t)x^{\sigma}(t) = \varphi(t, x(t)), & t \in \mathcal{I}^{\kappa}, \\ x(S) = x_0, \end{cases} \tag{1.1}$$

where $\mathcal{I} := [S, T] \cap \mathbb{T}$, $S, T \in \mathbb{T}$, $S < T$, $x_0 \in \mathbb{R}$, $\varphi \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T} \times \mathbb{R}, \mathbb{R})$, and $\psi \in \mathcal{R}(\mathbb{T}, \mathbb{R})$.

Motivated by Tisdell and Zaidi (2008); Bohner and Peterson (2001), in this paper, we shall find results related to existence of solutions of (1.1). Also, motivated by Tisdell and Zaidi (2008) and dos Santos (2015c), we shall focus on continuous dependence and convergence of solutions. The main tools that we use are fixed point theory and a dynamic version of Gronwall's inequality. Concerning an overview of time scales theory, we refer to Bohner and Peterson (2001) and Bohner and Peterson (2003). For related work, we refer to Anderson (2008), Abbas (2018), dos Santos and Silva (2013), Tikare and dos Santos (2020), dos Santos (2015a, 2015b), Tisdell and Zaidi (2008), Karpuz (2018) and Dai and Tisdell (2006). In Section 2, we give some preliminary results and definitions, while Section 3 contains an auxiliary result, an integral equation which is equivalent to (1.1). Section 4 contains our main results, three distinct existence theorems, a theorem on the continuous dependence of solutions, and a theorem on convergence of solutions.

2 Preliminaries

Here, some necessary definitions and results are recalled from the literature. We first collect the relevant material from time scales theory, then the pertinent material from fixed point theory.

2.1 Time scales

Definition 2.1 (See (Bohner and Peterson, 2001, Definition 1.58)): $\varphi \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$ provided φ is continuous at all right-dense points and its left-sided limits exist at all left-dense points.

Definition 2.2 (See (Karpuz, 2018, Definition 5)): We say $\varphi \in \mathrm{C_{rd}}(\mathbb{T} \times \mathbb{R}, \mathbb{R})$ provided $\varphi(\cdot, x) \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$ for all $x \in \mathbb{R}$ and $\varphi(t, \cdot) \in \mathrm{C}(\mathbb{R}, \mathbb{R})$ for all $t \in \mathbb{T}$.

Definition 2.3 (See (Bohner and Peterson, 2001, Definition 2.25)): We say $\psi \in \mathcal{R}(\mathbb{T}, \mathbb{R})$ provided $\psi \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$ and $1 + \mu(t)\psi(t) \neq 0$ for all $t \in \mathbb{T}^{\kappa}$.

Definition 2.4 (See (Bohner and Peterson, 2001, Definition 2.30)): For given $\psi \in \mathcal{R}$ and $t_0 \in \mathbb{T}$, the *exponential function* $e_{\psi}(\cdot, t_0)$ is defined to be the unique solution of

$$x^{\Delta} = \psi(t)x, \quad x(t_0) = 1.$$

Theorem 2.5 (See (Bohner and Peterson, 2001, Corollary 6.8)): *Let* $\varphi \in \mathrm{C_{rd}}(\mathcal{I}, \mathbb{R})$ *and* $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ *with* $\lambda_3 > 0$. *Then*

$$\varphi(t) \leq \lambda_1 + \lambda_2(t - S) + \lambda_3 \int_S^t \varphi(s)\Delta s \quad \text{for all} \quad t \in \mathcal{I}$$

implies

$$\varphi(t) \leq \left(\lambda_1 + \frac{\lambda_2}{\lambda_3}\right) e_{\lambda_3}(t, S) - \frac{\lambda_2}{\lambda_3} \quad \text{for all} \quad t \in \mathcal{I}.$$

2.2 Fixed point results

Theorem 2.6 (See Banach's fixed point theorem (Granas and Dugundji, 2003, Theorem 1.1 in §1.1)): If Ξ is a Banach space and $\Psi : \Xi \to \Xi$ is contractive, then Ψ has a unique fixed point $x^* \in \Xi$.

Theorem 2.7 (See (Granas and Dugundji, 2003, Corollary 1.2 in §1.1)): *If* Ξ *is a Banach space,* $\Psi : \{x \in \Xi : d(x, x_0) < r\} \to \Xi$ *is contractive with constant* $\alpha < 1$, *and*

$$d(\Psi(x_0), x_0) < (1 - \alpha)r,$$

then Ψ *has a fixed point.*

Definition 2.8 (Compact mapping (Pata, 2019, Definition 11.1)): A mapping between normed linear spaces is called *compact* provided bounded sets are mapped into relatively compact sets.

Definition 2.9 (Relatively compact set): A set M is called *relatively compact* provided its closure is compact.

Theorem 2.10 (Schaefer's fixed point theorem (Pata, 2019, Theorem 11.1))**:** *If Ξ is a Banach space, $\Psi : \Xi \to \Xi$ is a continuous and compact mapping, and*

$$\Gamma = \{x \in \Xi : \ x = \xi\Psi(x) \quad \text{for some} \quad \xi \in [0, 1]\}$$

is bounded, then Ψ has a fixed point in Ξ.

There are two forms of Arzelà–Ascoli's theorem, and they are stated as follows, see (Zhu and Wang, 2007, Lemma 4) and (Agarwal et al., 2003, Lemma 2.6).

Theorem 2.11: *A subset of $\mathrm{C}(\mathcal{I}, \mathbb{R})$ which is both equicontinuous and bounded is relatively compact.*

Theorem 2.12: *A sequence of functions which is both uniformly bounded and equicontinuous in $\mathcal{I}$ contains a uniformly convergent subsequence.*

3 Auxiliary result

The equivalence of (1.1) and a delta integral equation is given in the following lemma. The idea is the same as in (Bohner and Peterson, 2001, Theorem 2.74).

Lemma 3.1: *Let $S \in \mathbb{T}$, $x_0 \in \mathbb{R}$, $\psi \in \mathcal{R}(\mathcal{I}, \mathbb{R})$, and $\varphi \in \mathrm{C_{rd}}(\mathcal{I} \times \mathbb{R}, \mathbb{R})$. Then, x solves (1.1) iff*

$$x(t) = e_{\ominus\psi}(t, S)x_0 + \int_S^t e_{\ominus\psi}(t, s)\varphi(s, x(s))\Delta s. \tag{3.1}$$

Proof: First, assume $x : \mathcal{I} \to \mathbb{R}$ satisfies (1.1). Then,

$$\begin{aligned} e_\psi(t, S)\varphi(t, x(t)) &= e_\psi(t, S)x^\Delta(t) + e_\psi(t, S)\psi(t)x^\sigma(t) \\ &= (e_\psi(\cdot, S)x)^\Delta(t). \end{aligned}$$

Now integrating from S to $t \in \mathcal{I}$ and using the initial condition in (1.1), we obtain

$$\begin{aligned} \int_S^t e_\psi(s, S)\varphi(s, x(s))\Delta s &= e_\psi(t, S)x(t) - e_\psi(S, S)x(S) \\ &= e_\psi(t, S)x(t) - x_0. \end{aligned}$$

Multiplying now by $e_{\ominus\psi}(t, S)$ and using the time scales exponential rules, we get

$$\begin{aligned} x(t) - e_{\ominus\psi}(t, S)x_0 &= e_{\ominus\psi}(t, S)\int_S^t e_\psi(s, S)\varphi(s, x(s))\Delta s \\ &= \int_S^t e_{\ominus\psi}(t, S)e_\psi(s, S)\varphi(s, x(s))\Delta s \\ &= \int_S^t e_{\ominus\psi}(t, S)e_{\ominus\psi}(S, s)\varphi(s, x(s))\Delta s \\ &= \int_S^t e_{\ominus\psi}(t, s)\varphi(s, x(s))\Delta s. \end{aligned}$$

Hence, x satisfies equation (3.1).

Conversely, suppose that x satisfies equation (3.1). Then $x(S) = x_0$. Multiplying equation (3.1) by $e_\psi(t, S)$, we find

$$e_\psi(t,S)x(t) = x(S) + \int_S^t e_\psi(s,S)\varphi(s,x(s))\Delta s.$$

Taking the Δ-derivatives on both sides of this equation, we get

$$e_\psi(t,S)x^\Delta(t) + \psi(t)e_\psi(t,S)x^\sigma(t) = e_\psi(t,S)\varphi(t,x(t)).$$

Hence, x satisfies (1.1). □

For the main theorems presented below, we employ the notation

$$\begin{aligned} &\Xi := \mathrm{C}(\mathcal{I},\mathbb{R}),\\ &d(x,y) := \sup_{t\in\mathcal{I}} |x(t)-y(t)| \quad \text{for} \quad x,y\in\Xi,\\ &\|x\| := d(x,0) \quad \text{for} \quad x\in\Xi, \end{aligned}$$

and we introduce $\Psi : \Xi \to \Xi$ by

$$\Psi(x)(t) := e_{\ominus\psi}(t,S)x_0 + \int_S^t e_{\ominus\psi}(t,s)\varphi(s,x(s))\Delta s \quad \text{for} \quad t\in\mathcal{I},\ x\in\Xi.$$

Finally, we denote

$$E := \sup_{t,s\in\mathcal{I}} |e_{\ominus\psi}(t,s)| > 0.$$

4 Main results

In the following theorem, using Banach's fixed point theorem, Theorem 2.6, we establish existence of exactly one solution of (1.1).

Theorem 4.1: *Let $\varphi \in \mathrm{C_{rd}}(\mathcal{I}\times\mathbb{R},\mathbb{R})$. If there exists $L > 0$ with*

$$|\varphi(t,u) - \varphi(t,v)| \le L\,|u-v| \quad \textit{for all} \quad u,v\in\mathbb{R},\ t\in\mathcal{I}, \tag{4.1}$$

then (1.1) *has exactly one solution.*

Proof: From Lemma 3.1, fixed points of Ψ are solutions of (1.1). Therefore, we shall prove that the map Ψ has a unique fixed point $x \in \Xi$. For this, we show that Ψ is contractive. We have

$$\begin{aligned}|\Psi(x)(t) - \Psi(y)(t)| &\le \int_S^t |e_{\ominus\psi}(t,s)|\,|\varphi(s,x(s)) - \varphi(s,y(s))|\,\Delta s \\ &\le \int_S^t EL\,|x(s)-y(s)|\,\Delta s \\ &= EL\int_S^t |x(s)-y(s)|\,\Delta s \\ &\le EL\int_S^t d(x,y)\Delta s \\ &= ELd(x,y)(t-S),\end{aligned}$$

that is,

$$|\Psi(x)(t) - \Psi(y)(t)| \le ELh_1(t,S)d(x,y), \tag{4.2}$$

where $h_1(t,S) = t - S$. We claim that for $n \in \mathbb{N}$,

$$|\Psi^n(x)(t) - \Psi^n(y)(t)| \le E^n L^n h_n(t,S)d(x,y), \tag{4.3}$$

where h_n are defined recursively by $h_{n+1}(t,S) = \int_S^t h_n(s,S)\Delta s$, see (Bohner and Peterson, 2001, (1.9)). We shall prove this by induction on n. Clearly, from (4.2), (4.3) holds for $n = 1$. Assume that (4.3) holds for $n = k \in \mathbb{N}$, i.e.,

$$\left|\Psi^k(x)(t) - \Psi^k(y)(t)\right| \le E^k L^k h_k(t,S)d(x,y).$$

Now consider

$$\begin{aligned}&\left|\Psi^{k+1}(x)(t) - \Psi^{k+1}(y)(t)\right| = \left|\Psi(\Psi^k(x))(t) - \Psi(\Psi^k(y))(t)\right| \\ &\le \int_S^t |e_{\ominus\psi}(t,s)|\left|\varphi(s,\Psi^k(x)(s)) - \varphi(s,\Psi^k(y)(s))\right|\Delta s \\ &\le E\int_S^t \left|\varphi(s,\Psi^k(x)(s)) - \varphi(s,\Psi^k(y)(s))\right|\Delta s \\ &\le EL\int_S^t \left|\Psi^k(x)(s) - \Psi^k(y)(s)\right|\Delta s \\ &\le EL\int_S^t E^k L^k h_k(s,S)d(x,y)\Delta s \\ &= E^{k+1}L^{k+1}d(x,y)\int_S^t h_k(s,S)\Delta s \\ &= E^{k+1}L^{k+1}d(x,y)h_{k+1}(t,S).\end{aligned}$$

Hence, (4.3) holds for $n = k + 1$. Thus, (4.3) holds for all $n \in \mathbb{N}$. From (Bohner and Lutz, 2006, Theorem 4.1),

$$h_n(t,S) \le \frac{(T-S)^n}{n!} \quad \text{for all} \quad n \in \mathbb{N},\ t \in \mathcal{I}.$$

Using this in (4.3), we get

$$d(\Psi^n(x), \Psi^n(y)) \le \alpha_n d(x, y),$$

where

$$0 \le \alpha_n := \frac{(EL(T-S))^n}{n!} < 1 \quad \text{for sufficiently large} \quad n \in \mathbb{N}.$$

Thus, by (Granas and Dugundji, 2003, Result (A.1)), Ψ is contractive. Consequently, by Theorem 2.6, Ψ has exactly one fixed point. □

Remark 4.2: Under the conditions of Theorem 4.1, if x_i, $i \in \mathbb{N}$, are defined recursively by

$$x_{i+1}(t) = e_{\ominus\psi}(t, S)x_0 + \int_S^t e_{\ominus\psi}(t, s)\varphi(s, x_i(s))\Delta s,$$

then $x_i \to x^*$ uniformly on $\mathcal{I}$, where x^* is the only solution of (1.1).

In the following theorem, using the local version of Banach's fixed point theorem, Theorem 2.7, we find that a solution of (1.1) exists.

Theorem 4.3: *Let* $x_0 \in \mathbb{R}$, $\varphi \in \mathrm{C}_{\mathrm{rd}}(\mathcal{I} \times \mathbb{R}, \mathbb{R})$, *and*

$$M := \int_S^T |\varphi(t, x_0)| \,\Delta t.$$

If there exists

$$L \in \left[0, \frac{1}{E(T-S)}\right)$$

with

$$|\varphi(t, u) - \varphi(t, v)| \le L\,|u - v| \quad \textit{for all} \quad t \in \mathcal{I}$$

and all $u, v \in \mathbb{R}$ *satisfying* $|u - x_0| < r$ *and* $|v - x_0| < r$, *where*

$$r := \frac{(E+1)\,|x_0| + EM + 1}{1 - EL(T-S)},$$

then (1.1) *has at least one solution.*

Proof: We define

$$B := \{x \in \Xi : \, d(x, x_0) < r\},$$

where $x_0 \in \Xi$ is the constant function defined by $x_0(t) = x_0$ for all $t \in \mathbb{T}$. Let $x, y \in B$. Then

$$\begin{aligned} |\Psi(x)(t) - \Psi(y)(t)| &= \left| \int_S^t e_{\ominus\psi}(t,s)(\varphi(s,x(s)) - \varphi(s,y(s)))\Delta s \right| \\ &\le \int_S^T |e_{\ominus\psi}(t,s)|\,|\varphi(s,x(s)) - \varphi(s,y(s))|\,\Delta s \\ &\le E\int_S^T |\varphi(s,x(s)) - \varphi(s,y(s))|\,\Delta s \\ &\le E\int_S^T L\,|x(s) - y(s)|\,\Delta s \\ &\le ELd(x,y)\int_S^T \Delta s \\ &= EL(T-S)d(x,y) = \alpha d(x,y), \end{aligned}$$

so that

$$d(\Psi(x), \Psi(y)) \le \alpha d(x,y) \quad \text{with} \quad \alpha := EL(T-S) \in [0,1).$$

Hence, $\Psi : B \to \Xi$ is contractive with constant α. Next,

$$\begin{aligned} |\Psi(x_0)(t) - x_0(t)| &= \left| e_{\ominus\psi}(t,S)x_0 + \int_S^t e_{\ominus\psi}(t,S)\varphi(s,x_0)\Delta s - x_0 \right| \\ &= \left| (e_{\ominus\psi}(t,S) - 1)x_0 + \int_S^t e_{\ominus\psi}(t,S)\varphi(s,x_0)\Delta s \right| \\ &\le |e_{\ominus\psi}(t,S) - 1|\,|x_0| + \left| \int_S^t e_{\ominus\psi}(t,S)\varphi(s,x_0)\Delta s \right| \\ &\le (|e_{\ominus\psi}(t,S)| + 1)\,|x_0| + \int_S^t |e_{\ominus\psi}(t,S)|\,|\varphi(s,x_0)|\,\Delta s \\ &\le (E+1)\,|x_0| + E\int_S^t |\varphi(s,x_0)|\,\Delta s \\ &\le (E+1)\,|x_0| + EM \\ &< (E+1)\,|x_0| + EM + 1 = (1-\alpha)r, \end{aligned}$$

so that

$$d(\Psi(x_0), x_0) < (1-\alpha)r.$$

Consequently, by Theorem 2.7, Ψ has a fixed point, and this fixed point is a solution of (1.1). □

In the following theorem, using Schaefer's fixed point theorem, Theorem 2.10, we establish existence of at least one solution of (1.1).

Theorem 4.4: *Let $\varphi \in \mathrm{C_{rd}}(\mathcal{I} \times \mathbb{R}, \mathbb{R})$. If there exists $N > 0$ with*

$$|\varphi(t,u)| \le N(1 + |u|) \quad \textit{for all} \quad u \in \mathbb{R},\ t \in \mathcal{I}, \tag{4.4}$$

then (1.1) *has at least one solution.*

Proof: In a first step, we demonstrate that $\Psi : \Xi \to \Xi$ is continuous. Let $\{x_n : n \in \mathbb{N}\} \subset \Xi$ be such that $x_n \to x \in \Xi$ as $n \to \infty$. Then, for $t \in \mathcal{I}$, we find

$$\begin{aligned}|\Psi(x_n)(t) - \Psi(x)(t)| &= \left| \int_S^t e_{\ominus\psi}(t,s)(\varphi(s,x_n(s)) - \varphi(s,x(s)))\Delta s \right| \\ &\leq \int_S^t |e_{\ominus\psi}(t,s)|\, |\varphi(s,x_n(s)) - \varphi(s,x(s))|\, \Delta s \\ &\leq E \int_S^T |\varphi(s,x_n(s)) - \varphi(s,x(s))|\, \Delta s,\end{aligned}$$

so that

$$d(\Psi(x_n), \Psi(x)) \leq E \int_S^T |\varphi(s,x_n(s)) - \varphi(s,x(s))|\, \Delta s.$$

Due to the imposed condition on φ, we have $\Psi(x_n) \to \Psi(x)$ as $n \to \infty$. Hence, $\Psi : \Xi \to \Xi$ is indeed continuous. In a second step, we show that $\Psi : \Xi \to \Xi$ maps bounded sets into relatively compact sets. Let $\Omega \subset \Xi$ be bounded. Then there exists $K > 0$ with $\|x\| \leq K$ for all $x \in \Omega$. Now, let $x \in \Omega$. Then, for $t \in \mathcal{I}$, we obtain

$$\begin{aligned}|\Psi(x)(t)| &= \left| e_{\ominus\psi}(t,S)x_0 + \int_S^t e_{\ominus\psi}(t,s)\varphi(s,x(s))\Delta s \right| \\ &\leq |e_{\ominus\psi}(t,S)|\,|x_0| + \int_S^t |e_{\ominus\psi}(t,s)|\,|\varphi(s,x(s))|\,\Delta s \\ &\leq E\,|x_0| + E\int_S^t |\varphi(s,x(s))|\,\Delta s \\ &\leq E\,|x_0| + E\int_S^t N(1+|x(s)|)\Delta s \\ &\leq E\,|x_0| + E\int_S^t N(1+\|x\|)\Delta s \\ &\leq E\,|x_0| + EN(1+K)(T-S),\end{aligned}$$

so that

$$\|\Psi(x)\| \leq E\,|x_0| + EN(1+K)(T-S).$$

Therefore, $\Psi(\Omega)$ is bounded. Moreover, for $t_1, t_2 \in \mathcal{I}$ with $t_1 \leq t_2$, we get

$$\begin{aligned}&|\Psi(x)(t_2) - \Psi(x)(t_1)| \leq |e_{\ominus\psi}(t_2,S)x_0 - e_{\ominus\psi}(t_1,S)x_0| \\ &\quad + \left| \int_S^{t_2} e_{\ominus\psi}(t_2,s)\varphi(s,x(s))\Delta s - \int_S^{t_1} e_{\ominus\psi}(t_1,s)\varphi(s,x(s))\Delta s \right| \\ &\leq |(e_{\ominus\psi}(t_2,S) - e_{\ominus\psi}(t_1,S))\,x_0| + \left| \int_{t_1}^{t_2} e_{\ominus\psi}(t_2,s)\varphi(s,x(s))\Delta s \right| \\ &\quad + \left| \int_S^{t_1} (e_{\ominus\psi}(t_2,s) - e_{\ominus\psi}(t_1,s))\,\varphi(s,x(s))\Delta s \right|\end{aligned}$$

$$= |(e_{\ominus\psi}(t_2,S) - e_{\ominus\psi}(t_1,S))\, x_0| + \left|\int_{t_1}^{t_2} e_{\ominus\psi}(t_2,s)\varphi(s,x(s))\Delta s\right|$$
$$+ |e_{\ominus\psi}(t_2,S) - e_{\ominus\psi}(t_1,S)| \left|\int_{S}^{t_1} e_{\psi}(s,S)\varphi(s,x(s))\Delta s\right|$$
$$\leq |e_{\ominus\psi}(t_2,S) - e_{\ominus\psi}(t_1,S)|\,|x_0| + \int_{t_1}^{t_2} |e_{\ominus\psi}(t_2,s)|\,|\varphi(s,x(s))|\,\Delta s$$
$$+ |e_{\ominus\psi}(t_2,S) - e_{\ominus\psi}(t_1,S)| \int_{S}^{t_1} |e_{\psi}(s,S)|\,|\varphi(s,x(s))|\,\Delta s$$
$$\leq |e_{\ominus\psi}(t_2,S) - e_{\ominus\psi}(t_1,S)|\,|x_0| + \int_{t_1}^{t_2} EN(1+K)\Delta s$$
$$+ |e_{\ominus\psi}(t_2,S) - e_{\ominus\psi}(t_1,S)| \int_{S}^{T} EN(1+K)\Delta s$$
$$= |e_{\ominus\psi}(t_2,S) - e_{\ominus\psi}(t_1,S)|\,(|x_0| + EN(1+K)(T-S))$$
$$+ EN(1+K)(t_2-t_1).$$

If $t_1 \geq t_2$, then a similar calculation leads to the same result. Altogether, for any $t_1, t_2 \in \mathcal{I}$, we have

$$|x(t_2) - x(t_1)| \leq |e_{\ominus\psi}(t_2,S) - e_{\ominus\psi}(t_1,S)|\,(|x_0| + EN(1+K)(T-S)) + EN(1+K)(t_2-t_1).$$

As $t_2 - t_1 \to 0$, the right-hand side of this inequality tends to zero. Thus, $\Psi(\Omega)$ is equicontinuous. Now, since $\Psi(\Omega)$ is bounded and equicontinuous, using Theorem 2.11, $\Psi(\Omega)$ is relatively compact. Hence, $\Psi : \Xi \to \Xi$ indeed maps bounded sets into relatively compact sets. By Definition 2.8, $\Psi : \Xi \to \Xi$ is compact. In a third step, we show that

$$\Gamma = \{x \in \Xi : \ x = \xi\Psi(x) \quad \text{for some} \quad \xi \in [0,1]\}$$

is bounded. Let $x \in \Gamma$. Then, there is $\xi \in [0,1]$ with $x = \xi\Psi(x)$. For $t \in \mathcal{I}$, we get

$$|x(t)| = \xi\,|\Psi(x)(t)|$$
$$= \xi \left| e_{\ominus\psi}(t,a)x_0 + \int_S^t e_{\ominus\psi}(t,s)\varphi(s,x(s))\Delta s\right|$$
$$\leq |e_{\ominus\psi}(t,S)|\,|x_0| + \int_S^t |e_{\ominus\psi}(t,s)|\,|\varphi(s,x(s))|\,\Delta s$$
$$\leq E\,|x_0| + E\int_S^t |\varphi(s,x(s))|\,\Delta s$$
$$\leq E\,|x_0| + E\int_S^t N(1+|x(s)|)\Delta s$$
$$= E\,|x_0| + EN(t-S) + EN\int_S^t |x(s)|\,\Delta s$$

Employing Theorem 2.5 (note that $EN > 0$), for all $t \in \mathcal{I}$, we get

$$|x(t)| \leq (E|x_0|+1)e_{EN}(t,S) - 1 \leq (E|x_0|+1)e_{EN}(T,S) - 1,$$

where we used again $EN > 0$ in the last inequality. Hence,

$$\|x\| \le (E\,|x_0| + 1)e_{EN}(T, S) - 1,$$

so that Γ is indeed bounded. Consequently, by Theorem 2.10, Ψ has a fixed point. □

Remark 4.5: Under the conditions of Theorem 4.4, with the same calculations as in the proof of Theorem 4.4, we see that any solution $x \in \Xi$ (because $x = \Psi(x)$ according to Lemma 3.1) is bounded by

$$\|x\| \le (E\,|x_0| + 1)e_{EN}(T, S) - 1 \tag{4.5}$$

and satisfies, if $\|x\| \le K$,

$$\begin{aligned}|x(t_2) - x(t_1)| \le{}& |e_{\ominus\psi}(t_2, S) - e_{\ominus\psi}(t_1, S)|\,(|x_0| + EN(1 + K)(T - S))\\ &+ EN(1 + K)(t_2 - t_1) \quad \text{for all} \quad t_1, t_2 \in \mathcal{I}.\end{aligned} \tag{4.6}$$

Now, let us discuss some results concerning continuity and convergence of solutions of (1.1).

Theorem 4.6: *Let* $\varphi \in \mathrm{C_{rd}}(\mathcal{I} \times \mathbb{R}, \mathbb{R})$. *Assume* x *and* y *are solutions of* (1.1) *with* $x(S) = x_0$ *and* $y(S) = y_0$. *If there exists* $L > 0$ *such that* (4.1) *holds, then*

$$d(x, y) \le E\,|x_0 - y_0|\,e_{EL}(T, S).$$

Proof: Define $z := |x - y|$. Then, using Lemma 3.1, for $t \in \mathcal{I}$, we find

$$\begin{aligned}z(t) &= |x(t) - y(t)| = |\Psi(x)(t) - \Psi(y)(t)|\\ &\le |e_{\ominus\psi}(t, S)(x_0 - y_0)| + \left|\int_S^t e_{\ominus\psi}(t, s)(\varphi(s, x(s)) - \varphi(s, y(s)))\Delta s\right|\\ &\le E\,|x_0 - y_0| + E\int_S^t |\varphi(s, x(s)) - \varphi(s, y(s))|\,\Delta s\\ &\le E\,|x_0 - y_0| + EL\int_S^t |x(s) - y(s)|\,\Delta s\\ &= E\,|x_0 - y_0| + EL\int_S^t z(s)\Delta s.\end{aligned}$$

Employing Theorem 2.5 (note that $EL > 0$), for all $t \in \mathcal{I}$, we get

$$z(t) \le E\,|x_0 - y_0|\,e_{EL}(t, S) \le E\,|x_0 - y_0|\,e_{EL}(T, S),$$

where we used again $EL > 0$ in the last inequality. Hence,

$$\|z\| \le E\,|x_0 - y_0|\,e_{EL}(T, S),$$

which completes the proof. □

A convergence result of solutions of (1.1) is given in the final theorem.

Theorem 4.7: *Let $\varphi_k \in C_{rd}(\mathcal{I} \times \mathbb{R}, \mathbb{R})$, $k \in \mathbb{N}$. Suppose there exists $N > 0$ so that all φ_k satisfy* (4.4). *Assume x_k, $k \in \mathbb{N}$, solves*

$$\begin{cases} x^{\Delta}(t) + \psi(t)x^{\sigma}(t) = \varphi_k(t, x(t)), & t \in \mathcal{I}^{\kappa}, \\ x(S) = x_0^{(k)}, \end{cases} \tag{4.7}$$

where $x_0^{(k)} \to x_0 \in \mathbb{R}$ as $k \to \infty$. Let $\varphi \in C_{rd}(\mathcal{I} \times \mathbb{R}, \mathbb{R})$. If there exists $M > 0$ satisfying

$$|\varphi_k(t, u) - \varphi(t, v)| \le M\,|u - v| \quad \text{for all} \quad u, v \in \mathbb{R},\ t \in \mathcal{I},\ k \in \mathbb{N},$$

then there exists a subsequence $\{x_{k_j}\}$ that converges uniformly to a solution of (1.1).

Proof: For each $k \in \mathbb{N}$, φ_k satisfies the assumptions of Theorem 4.4. Thus, by (4.5)

$$\|x_k\| \le \left(E\left|x_0^{(k)}\right| + 1\right) e_{EN}(T, S) - 1.$$

Since $\{x_0^{(k)}\}$ is a real and convergent sequence, it is bounded, say by J, so that

$$\|x_k\| \le (EJ + 1)e_{EN}(T, S) - 1 =: K \quad \text{for all} \quad k \in \mathbb{N}.$$

Hence, $\{x_k\}$ is uniformly bounded. Next, by (4.6), for all $t_1, t_2 \in \mathcal{I}$, we have

$$\begin{aligned} |x_k(t_2) - x_k(t_1)| &\le |e_{\ominus\psi}(t_2, S) - e_{\ominus\psi}(t_1, S)| \left(\left|x_0^{(k)}\right| + EN(1 + K)(T - S)\right) \\ &\quad + EN(1 + K)(t_2 - t_1) \\ &\le |e_{\ominus\psi}(t_2, S) - e_{\ominus\psi}(t_1, S)| \,(J + EN(1 + K)(T - S)) \\ &\quad + EN(1 + K)(t_2 - t_1). \end{aligned}$$

Letting $t_2 - t_1 \to 0$ in the right-hand side of this inequality, we find that the limit is zero, independent of $k \in \mathbb{N}$. Therefore, $\{x_k\}$ is equicontinuous. Using Theorem 2.12, $\{x_k\}$ has a uniformly convergent subsequence, denoted once again by $\{x_k\}$. By Lemma 3.1,

$$x_k(t) = e_{\ominus\psi}(t, S)x_0^{(k)} + \int_S^t e_{\ominus\psi}(t, s)\varphi_k(s, x_k(s))\Delta s \tag{4.8}$$

holds for each $t \in \mathcal{I}$ and $k \in \mathbb{N}$. Letting now $k \to \infty$ in (4.8), we get

$$x(t) = e_{\ominus\psi}(t, S)x_0 + \int_S^t e_{\ominus\psi}(t, s)\varphi(s, x(s))\Delta s, \tag{4.9}$$

provided we can show that

$$\lim_{k\to\infty} \int_S^t e_{\ominus\psi}(t, s)\varphi_k(s, x_k(s))\Delta s = \int_S^t e_{\ominus\psi}(t, s)\varphi(s, x(s))\Delta s. \tag{4.10}$$

To show equation (4.10), we calculate

$$\begin{aligned}
&\left|\int_S^t e_{\ominus\psi}(t,s)\varphi_k(s,x_k(s))\Delta s - \int_S^t e_{\ominus\psi}(t,s)\varphi(s,x(s))\Delta s\right| \\
&\leq \int_S^t |e_{\ominus\psi}(t,s)|\,|\varphi_k(s,x_k(s)) - \varphi(s,x(s))|\,\Delta s \\
&\leq E\int_S^t |\varphi_k(s,x_k(s)) - \varphi(s,x(s))|\,\Delta s \\
&\leq E\int_S^t M\,|x_k(s) - x(s)|\,\Delta s \\
&\leq E\int_S^T M d(x_k, x)\Delta s \\
&= EM(T-S)d(x_k, x),
\end{aligned}$$

confirming equation (4.10). Thus, equation (4.9) holds. Employing Lemma 3.1 one last time completes the proof. □

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