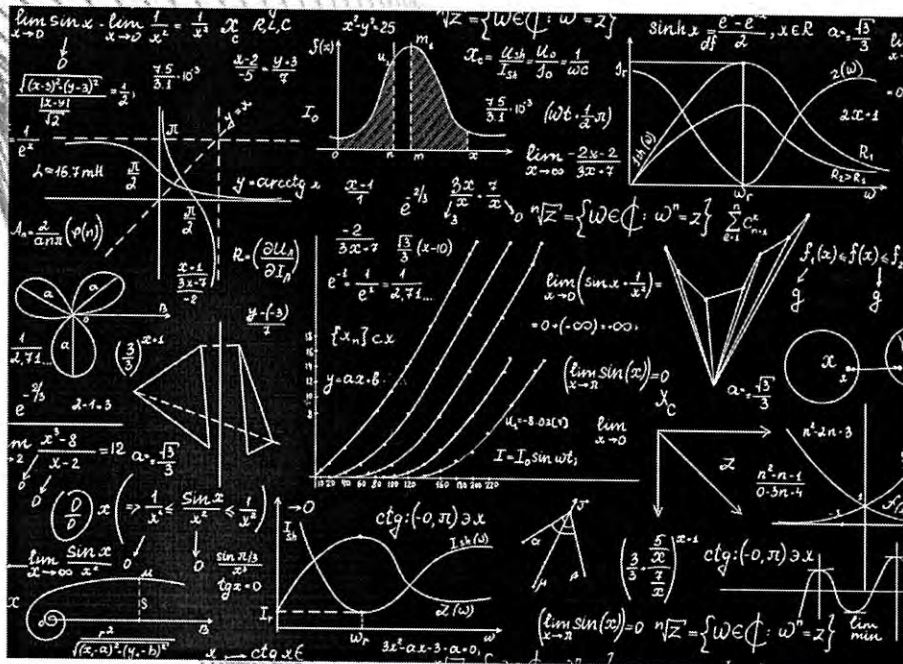


ADVANCES IN MATHEMATICAL ANALYSIS AND ITS APPLICATIONS



Edited by
BIPAN HAZARIKA
SANTANU ACHARJEE
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Advances in Mathematical Analysis and its Applications

Edited by
Bipan Hazarika
Santanu Acharjee
H. M. Srivastava



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Contents

Preface	xi
Editors	xv
Contributors	xvii
1 Some applications of double sequences	1
<i>Dragan Djurčić and Ljubiša D.R. Kočinac</i>	
1.1 Introduction	1
1.1.1 Double sequences	1
1.1.2 Selection principles	4
1.1.3 Asymptotic analysis	5
1.2 S_1 selection principle and double sequences	6
1.3 α_2 -selection principle and double sequences	9
1.4 Double sequences and the exponent of convergence	14
Bibliography	17
2 Convergent triple sequences and statistical cluster points	21
<i>Mehmet Gürdal and Mualla Birgül Huban</i>	
2.1 Introduction	21
2.2 Known definitions and properties	22
2.3 \mathcal{I}_3 -statistical cluster points of triple sequences	23
2.3.1 $\Gamma^{\mathcal{I}_3}$ -statistical convergence	28
2.4 Lacunary \mathcal{I}_3 -statistical cluster points	31
Bibliography	35
3 Relative uniform convergence of sequence of positive linear Functions	39
<i>Kshetrimayum Renubebeta Devi and Binod Chandra Tripathy</i>	
3.1 Introduction	39
3.2 Preliminaries and definitions	40
3.3 Relative uniform convergence of single sequence of functions	42
3.4 Statistical convergence of sequence	44
3.5 Double sequences	48
3.6 Relative uniform convergence of difference double sequence of positive linear functions	52
Bibliography	54

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v


Principal
Ramniranjan Jhunjhunwala College,
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4	Almost convergent sequence spaces defined by Nörlund matrix and generalized difference matrix	57
	<i>Kuldip Raj and Manisha Devi</i>	
4.1	Introduction and preliminaries	57
4.2	Main results	62
	Bibliography	68
5	Factorization of the infinite Hilbert and Cesàro operators	71
	<i>Hadi Roopaei</i>	
5.1	Introductions and preliminaries	71
5.2	Hilbert matrix	72
5.3	Hausdorff matrix	73
5.4	Cesàro matrix of order n	73
5.5	Copson matrix	75
5.6	Gamma matrix of order n	75
5.7	Factorization of the infinite Hilbert operator	76
5.7.1	Factorization of the Hilbert operator based on Cesàro operator	77
5.7.2	Factorization of the Hilbert operator based on gamma operator	81
5.7.3	Factorization of the Hilbert operator based on the generalized Cesàro operator	86
5.8	Factorization of the Cesàro operator	90
	Bibliography	93
6	On theorems of Galambos-Bojanić-Seneta type	95
	<i>Dragan Djurčić and Ljubiša D.R. Kočimac</i>	
6.1	Introduction	95
6.2	Known results	97
6.2.1	Classes ORV_s and ORV_f and their subclasses	97
6.2.2	Rapid and related variations	100
6.3	New result	106
	Bibliography	109
7	On the spaces of absolutely p-summable and bounded q-Euler difference sequences	113
	<i>Taja Yaying</i>	
7.1	Introduction	113
7.1.1	Euler matrix of order 1 and sequence spaces	114
7.1.2	Quantum calculus	115
7.2	q -Euler difference sequence spaces $e_p^q(\nabla)$ and $e_\infty^q(\nabla)$	117
7.3	Alpha-, beta-, and gamma-duals	119
7.4	Matrix transformations	122
	Bibliography	125

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Principal
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Ghatkopar (W), Mumbai-400086.

8	Approximation by the double sequence of LPO based on multivariable q-Lagrange polynomials	129
	<i>Behar Baxhaku and P. N. Agrawal</i>	
8.1	Introduction	129
8.2	Double sequence of $\mathfrak{R}_{n,q}^{\beta^{(1)}, \dots, \beta^{(r)}}(\cdot)(x)$	131
8.3	Approximation by using power series summability method (p.s.s.m)	132
8.3.1	Illustrative example	136
8.4	\mathcal{A} -statistical convergence of operators $\mathfrak{R}_{n_1, q_{n_1}}^{n_2, q_{n_2}}(\cdot)(\mathbf{x})$	139
8.4.1	Application of Theorem 8.4.4	143
8.5	\mathcal{A} -statistical convergence by GBS operators	146
	Bibliography	151
9	Results on interpolative Boyd-Wong contraction in quasi-partial b-metric space	155
	<i>Pragati Gautam and Swapnil Verma</i>	
9.1	Introduction and preliminaries	155
9.2	Main results	159
	Bibliography	165
10	Applications of differential transform method on some functional differential equations	169
	<i>Anil Kumar, Giriraj Methi, and Sanket Tikare</i>	
10.1	Introduction	169
10.2	Preliminaries	170
10.2.1	Definition of differential transform	171
10.2.2	Faà di Bruno's formula and Bell polynomials	171
10.2.3	Description of the method	173
10.2.4	Convergence results	174
10.2.5	Error estimate	175
10.3	Applications	175
10.3.1	Example 1	176
10.3.2	Example 2	178
10.3.3	Example 3	181
10.3.4	Example 4	184
10.3.5	Example 5	186
	Bibliography	188
11	Solvability of fractional integral equation via measure of noncompactness and shifting distance functions	191
	<i>Bhuban Chandra Deuri and Anupam Das</i>	
11.1	Introduction	191
11.1.1	Some notations	192
11.1.2	Measure of noncompactness	192
11.2	Main result	193

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Ghatkopar (W), Mumbai-400086.**

11.3 Application	196
Bibliography	201
12 Generalized fractional operators and inequalities integrals	205
<i>Juan E. Nápoles Valdés and Florencia Rabossi</i>	
12.1 Introduction	205
12.2 Integral inequalities with some integral operators	208
12.2.1 Generalized integral operators	208
12.2.2 Generalized fractional integral operators	212
12.2.3 Weighted integral operators	220
12.3 A general formulation of the notion of convex function	222
12.4 Integral inequalities and fractional derivatives	223
Bibliography	225
13 Exponentially biconvex functions and bivariational inequalities	229
<i>Muhammad Aslam Noor and Khalida Inayat Noor</i>	
13.1 Introduction	229
13.2 Preliminary results	231
13.3 Properties of exponentially biconvex functions	234
13.4 Bivariational inequalities	244
Bibliography	248
14 On a certain subclass of analytic functions defined by Bessel functions	251
<i>B. Venkateswarlu, P. Thirupathi Reddy, and Shashikala A</i>	
14.1 Introduction	251
14.2 Coefficient bounds	254
14.3 Neighborhood property	256
14.4 Partial sums	258
Bibliography	262
15 A note on meromorphic functions with positive coefficients defined by differential operator	265
<i>B. Venkateswarlu, P. Thirupathi Reddy, and Sujatha</i>	
15.1 Introduction	265
15.2 Coefficient inequality	268
15.3 Distortion theorem	269
15.4 Integral operators	271
15.5 Convex linear combinations and convolution properties	272
15.6 Neighborhood property	275
Bibliography	276

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Ghatkopar (W), Mumbai-400086.**

16 Sharp coefficient bounds and solution of the Fekete-Szegő problem for a certain subclass of bi-univalent functions associated with the Chebyshev polynomials	279
<i>Amol Bhausahab Patil</i>	
16.1 Introduction	279
16.1.1 Bi-univalent function	280
16.1.2 Subordination	281
16.1.3 Chebyshev polynomials	282
16.1.4 The function class $\mathcal{CH}_\Sigma(\lambda, \mu, x)$	282
16.2 Coefficient estimates for the class $\mathcal{CH}_\Sigma(\lambda, \mu, x)$	284
16.2.1 Some immediate consequences of the theorem	287
Bibliography	288
17 Some differential sandwich theorems involving a multiplier transformation and Ruscheweyh derivative	291
<i>Alb Lupaş Alina</i>	
17.1 Differential subordination and superordination	291
17.2 Strong differential subordination and superordination	300
Bibliography	308
18 A study on self similar, nonlinear and complex behavior of the spread of COVID-19 in India	311
<i>Dibakar Das, Sankalpa Chowdhury, Gourab Das, Anuska Chanda, Swapnesh Khamaru, and Koushik Ghosh</i>	
18.1 Introduction	312
18.2 On the importance of the tests performed	314
18.3 Theory	315
18.3.1 Calculation of moving averages	315
18.3.2 Calculation of Hurst exponent by finite variance scaling method	316
18.3.3 Estimation of fractal dimension by Higuchi's method	317
18.3.4 Multifractal analysis by multifractal detrended fluctuation analysis	318
18.3.5 Analysis for non-linearity using delay vector variance method	320
18.3.6 0-1 test for chaos detection	322
18.3.7 Mathematical aspects of self-organized criticality	323
18.4 Data	323
18.5 Results	324
Bibliography	330
Index	335

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Chapter 10

Applications of differential transform method on some functional differential equations

Anil Kumar

Giriraj Methi

Sanket Tikare

10.1	Introduction	169
10.2	Preliminaries	170
	10.2.1 Definition of differential transform	171
	10.2.2 Faà di Bruno's formula and Bell polynomials	171
	10.2.3 Description of the method	173
	10.2.4 Convergence results	174
	10.2.5 Error estimate	175
10.3	Applications	175
	10.3.1 Example 1	176
	10.3.2 Example 2	178
	10.3.3 Example 3	181
	10.3.4 Example 4	184
	10.3.5 Example 5	186
	Bibliography	188

10.1 Introduction

The functional differential equations with proportional delay were first studied by Ockendon and Taylor [18] in their work of collecting electric current for the pantograph of an electric locomotive, hence named pantograph equations. The investigation of these equations is important since they find applications in economic activities involving production, planning and decision

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making, number theory, biological phenomena, probability concepts applied on algebraic structures, electrodynamics, quantum mechanics, nautical science, and astrophysics, among others [2, 6, 14, 15]. Further, time delays are significant in engineering problems such as feedback loops equipped with sensors and actuators, the transmission of signals to remote center, in predictions and control systems, etc. [1, 9, 11, 16, 28].

Several analytical and numerical methods have been proposed by many researchers to study solutions of proportional delay differential equations (PDDEs) which include Adomian decomposition method (ADM) [5], Homotopy perturbation method (HPM) [26], Homotopy analysis method (HAM) [12], Variational iteration method (VIM) [8, 29], Taylor series [25], Runge-kutta method [30] and Collocation method [4]. Due to calculation of Adomian polynomials in ADM, evaluation of integrals in HPM, finding Lagrangian multipliers in VIM and discretization of variables and complex calculations in numerical methods make them unsuitable. We propose a simple approach involving the differential transformation in this chapter. The differential transformation has been introduced by G. Pukhov as the "Taylor transform" in 1976 and applied to the study of electrical circuits [19]. The differential transformation is closely related to Taylor expansion of real analytic functions. It has applications in solving different types of problems for all classes of differential equations (ordinary, partial, delayed, fractional, fuzzy etc.). The recent development and applications of DT are discussed in [3, 7, 13, 17, 20, 21, 23, 24] and references therein.

In the present chapter, the differential transformation is applied to solve proportional delay differential equations. The nonlinearity in the problems is addressed by using the partial ordinary Bell polynomials in the Faà di Bruno's formula. The results obtained by this technique are compared with analytical solutions. Detailed error analysis is provided. However, to the best of our knowledge, no researcher has applied the DTM using Bell polynomials on the practical problems discussed in the Section 10.3.

The chapter is organized as follows. In Section 10.2, we introduce the main idea and basic formulae of the differential transformation and provide necessary results to handle nonlinearity using partial ordinary Bell polynomials and discuss convergence results. Applications of the method are presented in Section 10.3.

10.2 Preliminaries

In this section, we discuss the main idea and basic formulae of the differential transformation as well as notation and results related to transformation of general nonlinear terms.

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10.2.1 Definition of differential transform

Let $w(v)$ be a real analytical function in a domain Ω and $v = v_0$ be an arbitrary point in Ω . Then, $w(v)$ can be expanded in a Taylor series in a neighborhood of the point $v = v_0$.

Definition 10.2.1. [22] The differential transformation of a real function $w(v)$ at a point $v_0 \in \mathbb{R}$ is $\mathcal{D}\{w(v)\}[v_0] = \{W(k)[v_0]\}_{k=0}^{\infty}$, where $W(k)[v_0]$, the differential transform of the k^{th} derivative of the function $w(v)$ at v_0 , is defined as

$$W(k)[v_0] = \frac{1}{k!} \left[\frac{d^k w(v)}{dv^k} \right]_{v=v_0}. \quad (10.1)$$

Definition 10.2.2. [22] The inverse differential transformation is given by

$$w(v) = \mathcal{D}^{-1} \left\{ \{W(k)[v_0]\}_{k=0}^{\infty} \right\} [v_0] = \sum_{k=0}^{\infty} W(k)[v_0] (v - v_0)^k. \quad (10.2)$$

Combining Definition 10.2.1 and Definition 10.2.2 give an expression of the function w in the form of the Taylor series:

$$w(v) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{d^k w(v)}{dv^k} \right]_{v=v_0} (v - v_0)^k. \quad (10.3)$$

In practical applications, the function $w(v)$ is expressed by a finite sum

$$w(v) = \sum_{k=0}^N W(k)[v_0] (v - v_0)^k, \quad (10.4)$$

since N can be chosen large enough to ensure that the effect of the remainder $\sum_{k=N+1}^{\infty} W(k)[v_0] (v - v_0)^k$ is arbitrarily small. The results which are used in this chapter are listed in Table 10.1 without proofs.

10.2.2 Faà di Bruno's formula and Bell polynomials

In the literature, it has been observed that differential transformation is not applied directly to nonlinear terms like w^n , $n \in \mathbb{N}$ or e^w . Authors [23] used Adomian polynomials to compute the differential transform of nonlinear terms. However, the differential transformation of nonlinear terms can be determined without calculating and evaluating symbolic derivatives by applying Faà di Bruno's formula to non-linear terms.

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TABLE 10.1: Formulae of the differential transform method

	Original function	Transformed function
1	$\frac{d^n w(v)}{dv^n}$	$(k+1)(k+2)(k+3)\dots(k+n)W(k+n)$
2	$w(v) = v^n$	$\delta(k-n) = \begin{cases} 1, & k=n \\ 0, & k \neq n \end{cases}$
3	$e^{\alpha v}$	$\frac{\alpha^k}{k!}$
4	$w_1(v)w_2(v)$	$\sum_{i=0}^k W_1(i)W_2(k-i)$
5	$w(\alpha v)$	$\alpha^k W(k)$
6	$w_1(\alpha_1 v)w_2(\alpha_2 v)$	$\sum_{i=0}^k (\alpha_1)^i (\alpha_2)^{k-i} W_1(i)W_2(k-i)$

Definition 10.2.3. [10] The partial ordinary Bell polynomials are the polynomials $\hat{B}_{k,l}(\hat{x}_1, \dots, \hat{x}_{k-l+1})$ in an infinite number of variables $\hat{x}_1, \hat{x}_2, \dots$ defined by the series expansion

$$\sum_{k \geq l} \hat{B}_{k,l}(\hat{x}_1, \dots, \hat{x}_{k-l+1})v^k = \left(\sum_{m \geq 1} \hat{x}_m v^m \right)^l, l = 0, 1, 2, \dots \quad (10.5)$$

Lemma 10.2.4. [22] The partial ordinary Bell polynomials $\hat{B}_{k,l}(\hat{x}_1, \dots, \hat{x}_{k-l+1}), l = 0, 1, 2, \dots, k \geq l$ satisfy the recurrence relation

$$\hat{B}_{k,l}(\hat{x}_1, \dots, \hat{x}_{k-l+1}) = \sum_{i=1}^{k-l+1} \frac{i \cdot l}{k} \hat{x}_i \hat{B}_{k-i, l-1}(\hat{x}_1, \dots, \hat{x}_{k-i-l+2}), \quad (10.6)$$

where $\hat{B}_{0,0} = 1$ and $\hat{B}_{k,0} = 0$ for $k \geq 1$.

Theorem 10.2.5. [22] Let g and f be real functions analytic near v_0 and $g(v_0)$, respectively, and let h be the composition $h(v) = (f \circ g)(v) = f(g(v))$. Denote $D\{g(v)\}[v_0] = \{G(k)\}_{k=0}^{\infty}$, $D\{f(v)\}[g(v_0)] = \{F(k)\}_{k=0}^{\infty}$ and $D\{(f \circ g)(v)\}[v_0] = \{H(k)\}_{k=0}^{\infty}$ the differential transformations of functions g, f , and h at $v_0, g(v_0)$, and v_0 , respectively. Then the numbers $H(k)$ in the sequence $\{H(k)\}_{k=0}^{\infty}$ satisfy the relations

$$\begin{aligned} H(0) &= F(0) \\ H(k) &= \sum_{l=1}^k F(l) \hat{B}_{k,l}(G(1), \dots, G(k-l+1)) \text{ for } k \geq 1. \end{aligned} \quad (10.7)$$

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10.2.3 Description of the method

Consider the proportional delay differential equation defined by

$$w^n(v) = f\left(v, w(v), w'(v), \dots, w^{(n-1)}(v), w\left(\frac{v}{\alpha_1}\right), w\left(\frac{v}{\alpha_2}\right), \dots, w\left(\frac{v}{\alpha_r}\right)\right), \quad (10.8)$$

where $\alpha_i \geq 1$ and $w^{(n)}$ is the n^{th} derivative of w with respect to v , for $n, r \in \mathbb{N}$.

Consider equation (10.8) subject to initial function $\phi(v) \in C^n([v^*, 0], \mathbb{R})$ where $v^* < 0$ such that

$$\phi(v_0) = w(v_0), \phi'(v_0) = w'(v_0), \dots, \phi^{(n-1)}(v_0) = w^{(n-1)}(v_0), \quad (10.9)$$

and subject to initial conditions

$$w(v_0) = w_0, w'(v_0) = w_1, \dots, w^{(n-1)}(v_0) = w_{n-1}. \quad (10.10)$$

Now equation (10.8) can be written in the form

$$L(w) + R(w) + M(v) = N(w). \quad (10.11)$$

The linear terms are split into L and R , where L is the highest order bounded linear operator and R is the remaining of the linear operators which are also bounded, M are continuous known functions satisfy the Lipschitz condition, and N are nonlinear terms.

Apply DT with Bell polynomial on equation (10.10)–(10.11),

$$\mathcal{D}(L(w)) + \mathcal{D}(R(w)) + \mathcal{D}(M(v)) = \mathcal{B}(N(w)), \quad (10.12)$$

$$\mathcal{D}(w(v_0)) = w_0, \mathcal{D}(w'(v_0)) = w_1, \dots, \mathcal{D}(w^{(n-1)}(v_0)) = w_{n-1}, \quad (10.13)$$

where \mathcal{D} is DT operator and \mathcal{B} is Bell polynomial operator.

From equation (10.12)–(10.13) we obtain following recursive relations,

$$\frac{(k+n)!}{k!} W(k+n) + W(k) + M(k) = \mathcal{B}(N(w)), \quad (10.14)$$

$$W(0) = w_0, W(1) = w_1, \dots, W(n-1) = \frac{1}{(n-1)!} w_{(n-1)}. \quad (10.15)$$

If $N(w) = H(v) = f(g(v))$ then nonlinear Bell polynomial operator \mathcal{B} are defined by Theorem (10.2.5) as

$$H(0) = F(0),$$

$$H(k) = \sum_{l=1}^k F(l) \cdot \hat{B}_{k,l}(G(1), \dots, G(k-l+1)) \text{ for } k \geq 1. \quad (10.16)$$

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We can easily obtain different components using equations (10.14)–(10.16), and then using inverse transformation, we obtain approximate solution in the form of Taylor series

$$w(v) = \sum_{k=0}^{\infty} W(k)(v - v_0)^k. \quad (10.17)$$

10.2.4 Convergence results

In this section, we discuss the convergence results used in this chapter. The proof is taken from [23, 27].

Theorem 10.2.6. *Let f be an analytical function in $[0, T] \times \mathbb{R}^n \times \mathbb{R}^n$. Assume that problem (10.8) has unique solution in some interval $[0, T]$. Let $B_k = W(k)v^k$. If there exist a constant δ , $0 < \delta < 1$, $k_0 \in \mathbb{N}$ such that $\|(B_{k+1}(v))\| \leq \alpha\|(B_k(v))\|$ for all, then the series converges to the unique solution on the interval $J = [0, \alpha]$, $\alpha \leq T$.*

Proof. Let $C^n(J)$ be a Banach space of vector-valued functions $h(v) = (h_1(v), \dots, h_p(v))^T$ with continuous derivatives up to order n and norm $\|h(v)\| = \max_{i=1, \dots, p} \max_{j=0, \dots, n} \max_{v \in J} |h_i^{(j)}(v)|$.

Assume $S_l = \sum_{k=0}^l B_k(v)$. Now it is sufficient to prove that sequence $\{S_l\}$ is a Cauchy sequence in $C^n(J)$.

Due to

$$\|S_{l+1} - S_l\| = \|B_{l+1}(v)\| \leq \delta \|B_l(v)\| \leq \dots \delta^{l-n_0+1} \|B_{n_0}(v)\|$$

for every $l, m \in \mathbb{N}$, $l \geq m > n_0$, we get

$$\begin{aligned} \|S_l - S_m\| &= \left\| \sum_{j=m}^{l-1} (S_{j+1} - S_j) \right\| \leq \sum_{j=m}^{l-1} \|(S_{j+1} - S_j)\| \leq \sum_{j=m}^{l-1} \delta^{j-n_0+1} \|B_{n_0}(v)\| \\ &= \delta^{m-n_0+1} (1 + \delta + \delta^2 + \dots + \delta^{l-m+1}) \|B_{n_0}(v)\| \\ &= \frac{1 - \delta^{l-m}}{1 - \delta} \delta^{m-n_0+1} \|B_{n_0}(v)\|. \end{aligned} \quad (10.18)$$

Since $0 < \delta < 1$, it follows that

$$\lim_{l, m \rightarrow \infty} \|S_l - S_m\| = 0.$$

Therefore $\{S_l\}$ is a Cauchy sequence in $C^n(J)$ and the proof is complete. \square

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Theorem 10.2.7. Suppose that the assumptions of Theorem (10.2.6) are valid. Then for the truncated series $\sum_{k=0}^m B_k(v)$, the following error estimate holds

$$\left\| w(v) - \sum_{k=0}^m B_k(v) \right\| \leq \frac{1}{1-\delta} \delta^{m-m_0+1} \max_{i=1, \dots, p} \max_{j=0, \dots, n} \left| \frac{m_0!}{(m_0-j)!} W_i(m_0) \alpha^{m_0-j} \right|$$

for any $m \geq 0$, $m \geq m_0$.

Proof. Without loss of generality, we can choose $m_0 \geq n$, where n is the order of the system (10.8). From inequality (10.18) we have

$$\begin{aligned} \|S_l - S_m\| &\leq \frac{1 - \delta^{l-m}}{1 - \delta} \delta^{m-m_0+1} \|B_{m_0}(v)\| \\ &= \frac{1 - \delta^{l-m}}{1 - \delta} \delta^{m-m_0+1} \max_{i=1, \dots, p} \max_{j=0, \dots, n} \left| \frac{m_0!}{(m_0-j)!} W_i(m_0) \alpha^{m_0-j} \right| \end{aligned} \quad (10.19)$$

for $l \geq m \geq m_0$.

From $0 < \delta < 1$ it follows $(1 - \delta^{l-m}) < 1$. Hence, inequality (10.19) can be reduced to

$$\frac{1}{1-\delta} \delta^{m-m_0+1} \max_{i=1, \dots, p} \max_{j=0, \dots, n} \left| \frac{m_0!}{(m_0-j)!} W_i(m_0) \alpha^{m_0-j} \right|$$

Hence, we use the fact that $l \rightarrow \infty$, $S_l \rightarrow w(v)$, and so proof is complete. \square

10.2.5 Error estimate

For comparison, absolute error and maximum absolute error are computed and defined as

$$\begin{aligned} E_N(v) &:= |w(v) - w_N(v)|, \\ E_{N,\infty} &:= \max_{0 \leq v \leq 1} E_N(v), \end{aligned}$$

where $w(v)$ is the exact solution and $w_N(v)$ is the truncated series solution with degree N . Furthermore, the relative error between exact and approximate solution is defined by $R_N(v) := \frac{E_N(v)}{|w(v)|}$.

10.3 Applications

Five examples are discussed to show reliability and accuracy of the presented method. The MATHEMATICA software version 11 has been used for numerical computations.

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10.3.1 Example 1

Consider the following linear proportional DDE

$$w''(v) = 2e^{\frac{2v}{3}} w\left(\frac{v}{3}\right) - w'(v) - w\left(\frac{v}{2}\right) + e^{v/2}, \quad 0 \leq v \leq 1, \quad (10.20)$$

with initial conditions

$$w(0) = w'(0) = 1. \quad (10.21)$$

The exact solution is given by

$$w(v) = e^v. \quad (10.22)$$

Applying differential transform to equations (10.20)–(10.21), we obtain the following recursive relation

$$W(k+2) = \frac{1}{(k+1)(k+2)} \left(2 \sum_{r=0}^k \left(\frac{1}{3}\right)^k \frac{2^{k-r}}{(k-r)!} W(r) - (k+1)W(k+1) - \left(\frac{1}{2}\right)^k W(k) + \left(\frac{1}{2}\right)^k \frac{1}{k!} \right), \quad (10.23)$$

$$W(0) = W(1) = 1. \quad (10.24)$$

Using equations (10.23)–(10.24), we obtain the following components,

$$\begin{aligned} k=0, \quad W(2) &= \frac{1}{2} (2W(0) - W(1) - W(0) + 1) = \frac{1}{2!}, \\ k=1, \quad W(3) &= \frac{1}{6} \left(2 \left(\frac{2}{3}W(0) + \frac{1}{3}W(1) \right) - 2W(2) - \frac{1}{2}W(1) + \frac{1}{2} \right) = \frac{1}{3!}, \\ k=2, \quad W(4) &= \frac{1}{4!}, \dots, \text{ and so on.} \end{aligned} \quad (10.25)$$

Now, using equation (10.4), the series solution is given by

$$w(v) = 1 + v + \frac{1}{2!}v^2 + \frac{1}{3!}v^3 + \dots, \quad (10.26)$$

which converges to the exact solution given by equation (10.22). The approximate solution for $N = 12$ is compared with the analytical solution in Table 10.2, where N represents a number of terms considered. Table 10.3 lists the maximal absolute error of approximate results obtained by the present method for $N = 5, 10$, and 15 . Figure 10.1 depicts absolute errors for the numerical solutions for $N = 5, 10$, and 15 . From these results, it is clear that absolute errors and maximal absolute errors all decline systematically with the increase in N .

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TABLE 10.2: Comparison of numerical solution $w(v)$ with exact solution when $N = 12$ for Example 1

v	$w(v)$	$w_N(v)$	$R_N(v)$
0.1	1.105170918	1.105170918	2.0E-16
0.2	1.221402758	1.221402758	2.0E-16
0.3	1.349858807	1.349858807	1.6E-16
0.4	1.491824697	1.491824697	7.4E-16
0.5	1.648721270	1.648721270	1.2E-14
0.6	1.822118800	1.822118800	1.2E-13
0.7	2.013752707	2.013752707	8.1E-13
0.8	2.225540928	2.225540928	4.2E-12
0.9	2.459603111	2.459603111	1.7E-11
1.0	2.718281828	2.718281828	6.3E-11

TABLE 10.3: Maximum absolute errors for w of Example 1

N	$E_{N,\infty}$
5	9.9E-03
10	3.0E-07
15	8.1E-13

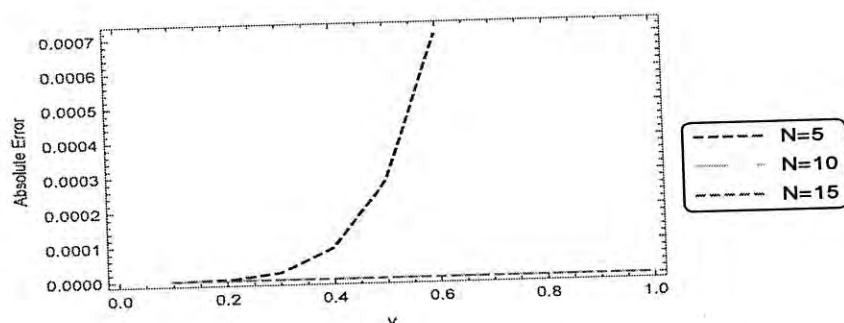


FIGURE 10.1: Absolute errors for $N = 5, 10,$ and 15 of Example 1.

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10.3.2 Example 2

Consider the following linear system of proportional DDEs

$$u'(v) = u(v) + 2e^v w\left(\frac{v}{3}\right) - e^{2v}, \quad (10.27)$$

$$w'(v) = 2w(v) + e^{2v} u\left(\frac{v}{2}\right), \quad 0 \leq v \leq 1, \quad (10.28)$$

with initial conditions

$$u(0) = w(0) = 1. \quad (10.29)$$

The exact solution is given by

$$u(v) = e^{2v}, \quad w(v) = e^{3v}. \quad (10.30)$$

Employing the differential transform to equations (10.27)–(10.29), we obtain the following recursive relation

$$U(k+1) = \frac{1}{(k+1)} \left(U(k) + 2 \sum_{r=0}^k \frac{1}{r!} \left(\frac{1}{3}\right)^{k-r} W(k-r) - \frac{2^k}{k!} \right), \quad (10.31)$$

$$W(k+1) = \frac{1}{(k+1)} \left(2W(k) + \sum_{r=0}^k \frac{2^r}{r!} \left(\frac{1}{2}\right)^{k-r} U(k-r) \right), \quad (10.32)$$

$$U(0) = W(0) = 1. \quad (10.33)$$

Using equations (10.31)–(10.33), we obtain the following components,

$$k = 0, \quad U(1) = U(0) + 2W(0) - 1 = 2,$$

$$W(1) = 2W(0) + U(0) = 3,$$

$$k = 1, \quad U(2) = \frac{1}{2} \left(U(1) + 2 \left(\frac{1}{3} W(1) + W(0) \right) - 2 \right) = \frac{2^2}{2!},$$

$$W(2) = \frac{1}{2} \left(2W(1) + \frac{1}{2} U(1) + 2U(0) \right) = \frac{3^2}{2!},$$

$$k = 2, \quad U(3) = \frac{2^3}{3!},$$

$$W(3) = \frac{3^3}{3!}, \dots, \text{ and so on.} \quad (10.34)$$

Now, with the help of equation (10.4), the series solution is given by

$$u(v) = 1 + 2v + \frac{2^2}{2!}v^2 + \frac{2^3}{3!}v^3 + \dots, \quad (10.35)$$

$$w(v) = 1 + 3v + \frac{3^2}{2!}v^2 + \frac{3^3}{3!}v^3 + \dots, \quad (10.36)$$

which converges to the exact solution given by equation (10.30).

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The approximate solution for $N = 12$ is compared with the analytical solution in Tables (10.4)–(10.5), where N represents a number of terms considered. Table (10.6) lists the maximal absolute error of approximate results obtained by the present method for $N = 5, 10$, and 15 . Figure 10.2 depicts absolute errors for the numerical solutions for $N = 5, 10$, and 15 . From these results, it is clear that absolute errors and maximal absolute errors all decline systematically with the increase in N .

TABLE 10.4: Comparison of numerical solution $u(v)$ with exact solution when $N = 12$ for Example 2

v	$u(v)$	$u_N(v)$	$R_N(v)$
0.1	1.221402758	1.221402758	0
0.2	1.491824698	1.491824698	2.4E-14
0.3	1.822118800	1.822118800	2.6E-12
0.4	2.225540928	2.225540928	6.8E-11
0.5	2.718281828	2.718281826	8.3E-10
0.6	3.320116923	3.320116902	6.1E-09
0.7	4.055199967	4.055199834	3.2E-08
0.8	4.953032424	4.953031755	1.3E-07
0.9	6.049647464	6.049644666	4.6E-07
1.0	7.389056099	7.389046016	1.3E-06

TABLE 10.5: Comparison of numerical solution $w(v)$ with exact solution when $N = 12$ for Example 2

v	$w(v)$	$w_N(v)$	$R_N(v)$
0.1	1.349858808	1.349858808	9.8E-16
0.2	1.8221188	1.8221188	2.6E-12
0.3	2.459603111	2.459603111	2.5E-10
0.4	3.320116923	3.320116902	6.1E-09
0.5	4.48168907	4.481688764	6.8E-08
0.6	6.049647464	6.049644666	4.6E-07
0.7	8.166169913	8.166151643	2.2E-06
0.8	11.02317638	11.0230832	8.4E-06
0.9	14.87973172	14.87933803	2.6E-05
1.0	20.08553692	20.08410308	7.1E-05

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TABLE 10.6: Maximum absolute errors for u and w of example 2

N	$E_{N,\infty}$ for u	$E_{N,\infty}$ for w
5	3.8E-01	3.7E-00
10	3.4E-04	2.2E-02
15	2.8E-08	1.3E-05

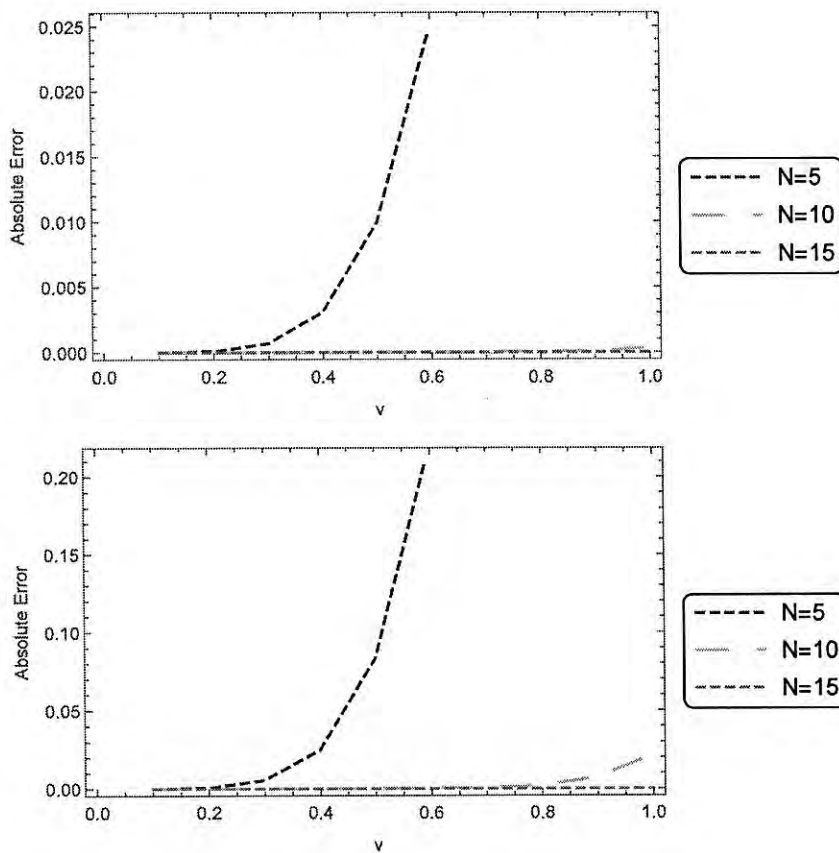


FIGURE 10.2: Absolute errors for u and w for $N = 5, 10,$ and 15 of Example 2.

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10.3.3 Example 3

Consider the following nonlinear system of proportional DDEs

$$u'(v) = u\left(\frac{v}{2}\right)w\left(\frac{v}{4}\right) - u(v) - u\left(\frac{v}{4}\right), \quad (10.37)$$

$$w'(v) = u\left(\frac{v}{2}\right)w(v) - w(v) - w\left(\frac{v}{2}\right), \quad 0 \leq v \leq 1, \quad (10.38)$$

with initial conditions

$$u(0) = w(0) = 1. \quad (10.39)$$

The exact solution is given by

$$u(v) = e^{-v}, \quad w(v) = e^v. \quad (10.40)$$

Applying differential transform to equations (10.37)–(10.39), we obtain the following recursive relation

$$U(k+1) = \frac{1}{(k+1)} \left(\sum_{r=0}^k \left(\frac{1}{2}\right)^{2k-r} U(r)W(k-r) - U(k) - \left(\frac{1}{4}\right)^k U(k) \right), \quad (10.41)$$

$$W(k+1) = \frac{1}{(k+1)} \left(\sum_{r=0}^k \left(\frac{1}{2}\right)^r U(r)W(k-r) + W(k) - \left(\frac{1}{2}\right)^k W(k) \right), \quad (10.42)$$

$$U(0) = W(0) = 1. \quad (10.43)$$

Using equations (10.41)–(10.43), we obtain the following components,

$$k=0, \quad U(1) = U(0)W(0) - U(0) - U(0) = -1,$$

$$W(1) = U(0)W(0) + W(0) - W(0) = 1,$$

$$k=1, \quad U(2) = \frac{1}{2} \left(\frac{1}{4}U(0)W(1) + \frac{1}{2}U(1)W(0) - U(1) - \frac{1}{4}U(1) \right) = -\frac{1}{2!},$$

$$W(2) = \frac{1}{2} \left(U(0)W(1) + \frac{1}{2}U(1)W(0) + W(1) - \frac{1}{2}U(1) \right) = \frac{1}{2!},$$

$$k=2, \quad U(3) = -\frac{1}{3!},$$

$$W(3) = \frac{1}{3!}, \dots, \text{ and so on.} \quad (10.44)$$

Now, with the help of equation (10.4), the series solution is given by

$$u(v) = 1 - v + \frac{1}{2!}v^2 - \frac{1}{3!}v^3 + \dots, \quad (10.45)$$

$$w(v) = 1 + v + \frac{1}{2!}v^2 + \frac{1}{3!}v^3 + \dots, \quad (10.46)$$

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which converges to the exact solution given by equation (10.40).

The approximate solution for $N = 12$ is compared with the analytical solution in Tables (10.7)–(10.8), where N represents a number of terms considered. Table 10.9 lists the maximal absolute error of approximate results obtained by the present method for $N = 5, 10,$ and 15 . Figure 10.3 depicts absolute errors for the numerical solutions for $N = 5, 10,$ and 15 . From these results, it is clear that absolute errors and maximal absolute errors all decline systematically with the increase in N .

TABLE 10.7: Comparison of numerical solution $u(v)$ with exact solution when $N = 12$ for Example 3

v	$u(v)$	$u_N(v)$	$R_N(v)$
0.0	1.000000000	1.000000000	0
0.1	0.904837418	0.904837418	1.2E-16
0.2	0.818730753	0.818730753	1.3E-16
0.3	0.740818220	0.740818220	1.3E-16
0.4	0.670320046	0.670320046	1.4E-15
0.5	0.606530659	0.606530659	3.1E-14
0.6	0.548811636	0.548811636	3.6E-13
0.7	0.496585303	0.496585303	2.9E-12
0.8	0.449328964	0.449328964	1.8E-11
0.9	0.406569659	0.406569659	9.4E-11
1.0	0.367879441	0.367879441	4.0E-10

TABLE 10.8: Comparison of numerical solution $w(v)$ with exact solution when $N = 12$ for Example 3

v	$w(v)$	$w_N(v)$	$R_N(v)$
0.0	1.000000000	1.000000000	0
0.1	1.105170918	1.105170918	2.0E-16
0.2	1.221402758	1.221402758	2.0E-16
0.3	1.349858807	1.349858807	1.6E-16
0.4	1.491824697	1.491824697	7.4E-16
0.5	1.648721270	1.648721270	1.2E-14
0.6	1.822118800	1.822118800	1.2E-13
0.7	2.013752707	2.013752707	8.1E-13
0.8	2.225540928	2.225540928	4.2E-12
0.9	2.459603111	2.459603111	1.7E-11
1.0	2.718281828	2.718281828	6.3E-11

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TABLE 10.9: Maximum absolute errors for u and w of Example 3

N	$E_{N,\infty}$ for u	$E_{N,\infty}$ for w
5	1.2E-03	9.9E-03
10	2.3E-08	3.0E-07
15	7.1E-13	8.1E-13

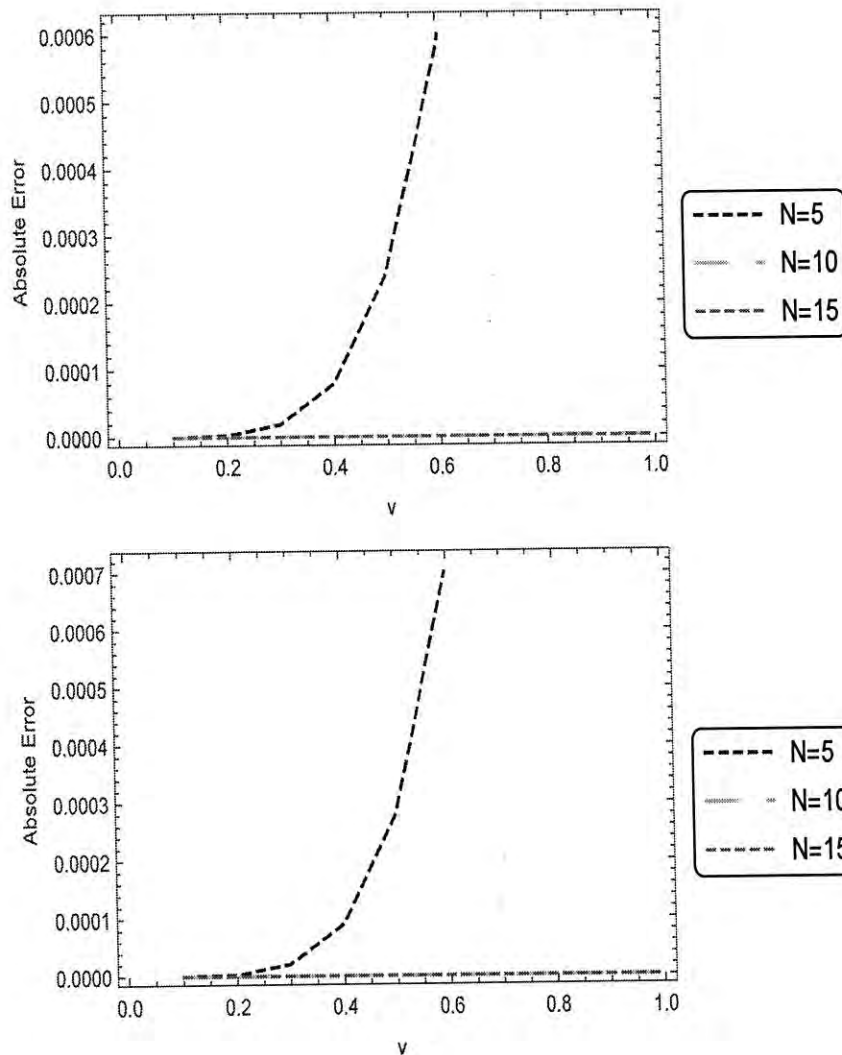


FIGURE 10.3: Absolute errors for u and w for $N = 5, 10,$ and 15 of Example 3.

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10.3.4 Example 4

Consider the following nonlinear proportional DDE

$$(1+v)w'(v) = (1+v)e^{w(\frac{v}{2})} - \frac{1}{2}v^2 - \frac{3}{2}v, \quad 0 \leq v \leq 1, \quad (10.47)$$

with initial condition

$$w(0) = 0. \quad (10.48)$$

The exact solution is given by

$$w(v) = \log(1+v). \quad (10.49)$$

Denote $h(v) = f(g(v))$, where $g(v) = w(\frac{v}{2})$ and $f(x) = e^x$.
Differential transform of $f(x)$ is represented by $F(k)$ then

$$F(k) = \frac{1}{k!}, \quad (10.50)$$

and differential transform of $h(v)$ is represented by $H(k)$ then using theorem (10.2.5), we have $H(0) = 1$ and

$$H(k) = \left(\frac{1}{2}\right)^k \sum_{l=1}^k F(l) \hat{B}_{k,l}(W(1), \dots, W(k-l+1)) \text{ for } k \geq 1.$$

Applying differential transform to equations (10.47)–(10.48), we obtain the following recursive relation

$$W(k+1) = \frac{1}{(k+1)} \left(- \sum_{r=0}^k \delta(r-1)(k-r+1)W(k-r+1) + H(k) + \sum_{r=0}^k \delta(r-1)H(k-r) - \frac{1}{2}\delta(k-2) - \frac{3}{2}\delta(k-1) \right) \quad (10.51)$$

$$W(0) = 0. \quad (10.52)$$

Using equations (10.50)–(10.52) we obtain following components,

$$k=0, W(1) = H(0) = 1,$$

$$k=1, H(1) = \left(\frac{1}{2}\right) F(1) \hat{B}_{1,1}(W(1)) = \frac{1}{2} F(1) W(1) = \frac{1}{2},$$

$$W(2) = \frac{1}{2} \left(-W(1) + H(1) + H(0) - \frac{3}{2} \right) = -\frac{1}{2},$$

$$k=2, H(2) = \left(\frac{1}{2}\right)^2 \left(\sum_{l=1}^2 F(l) \hat{B}_{2,1}(W(1), W(2)) \right)$$

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$$\begin{aligned}
 &= \frac{1}{4} (F(1)W(2) + F(2)W^2(1)) = 0, \\
 W(3) &= \frac{1}{3} \left(-2W(2) + H(2) + H(1) - \frac{1}{2} \right) \\
 &= \frac{1}{3}, \dots \text{ and so on.}
 \end{aligned} \tag{10.53}$$

Now, with the help of equation (10.4), the series solution is given by

$$w(v) = v - \frac{1}{2}v^2 + \frac{1}{3}v^3 - \dots, \tag{10.54}$$

which converges to the exact solution given by equation (10.49).

The approximate solution for $N = 15$ is compared with the analytical solution in Table 10.10, where N represents a number of terms considered. Table 10.11 lists the maximal absolute error of approximate results obtained by the present method for $N = 5, 10$, and 15 . Figure 10.4 depicts absolute errors for the numerical solutions for $N = 5, 10$, and 15 .

TABLE 10.10: Comparison of numerical solution $w(v)$ with exact solution when $N = 15$ for Example 4

v	$w(v)$	$w_N(v)$	$R_N(v)$
0.1	0.0953101798	0.0953101798	7.2E-16
0.2	0.1823215568	0.1823215568	1.8E-12
0.3	0.2623642645	0.2623642647	7.9E-10
0.4	0.3364722366	0.3364722561	5.7E-08
0.5	0.4054651081	0.4054657568	1.5E-06
0.6	0.4700036292	0.4700149031	3.9E-05
0.7	0.5306282511	0.5307535353	2.3E-04
0.8	0.5877866649	0.5887909192	1.7E-03
0.9	0.6418538862	0.6481288691	9.7E-03
1.0	0.6931471806	0.7253718504	4.6E-02

TABLE 10.11: Maximum absolute errors for w of Example 4

N	$E_{N,\infty}$
5	9.0E-02
10	4.7E-02
15	3.2E-02

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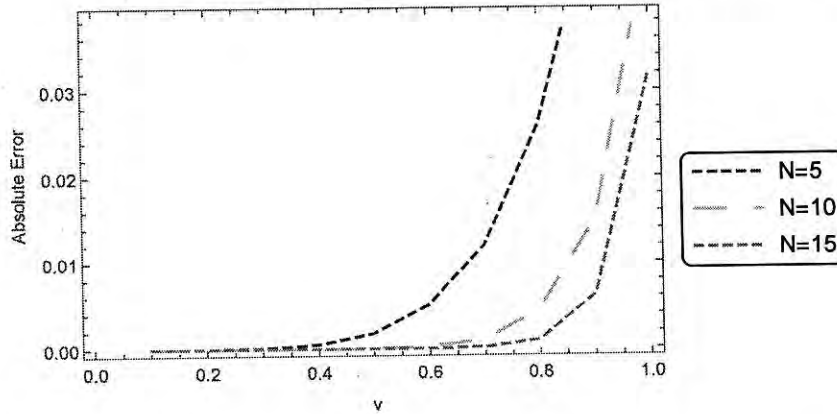


FIGURE 10.4: Absolute errors for $N = 5, 10,$ and 15 of Example 4.

10.3.5 Example 5

Consider the following nonlinear proportional DDE

$$w'(v) = 4w' \left(\frac{v}{2} \right) \sqrt{1 - w^2 \left(\frac{v}{2} \right)} - 2, \quad 0 \leq v \leq 1, \quad (10.55)$$

with initial condition

$$w(0) = 0. \quad (10.56)$$

The exact solution is given by

$$w(v) = \sin(2v). \quad (10.57)$$

Denote $h(v) = f(g(v))$, where $g(v) = w\left(\frac{v}{2}\right)$ and $f(x) = \sqrt{1 - x^2}$. Differential transform of $f(x)$ is represented by $F(k)$ then

$$F(k) = \begin{cases} \binom{1/2}{k} (-1)^k, & k \text{ is even} \\ 0, & k \text{ is odd,} \end{cases} \quad (10.58)$$

and differential transform of $h(t)$ is represented by $H(k)$ then using theorem (10.2.5), we have $H(0) = 1$ and

$$H(k) = \left(\frac{1}{2} \right)^k \sum_{l=1}^k F(l) \hat{B}_{k,l}(W(1), \dots, W(k-l+1)) \quad \text{for } k \geq 1.$$

Applying differential transform to equations (10.55)–(10.56), we obtain the following recursive relation

$$W(k+1) = \frac{4}{(k+1)} \left(\sum_{r=0}^k \left(\frac{1}{2} \right)^{k-r+1} (k-r+1) H(r) W(k-r+1) - 2\delta(k) \right) \quad (10.59)$$

$$W(0) = 0. \quad (10.60)$$

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Using equations (10.58)–(10.60) we obtain following components,

$$\begin{aligned}
 k = 0, W(1) &= \frac{4}{2}H(0)W(1) - 2 = 2, \\
 k = 1, H(1) &= \left(\frac{1}{2}\right)F(1)\hat{B}_{1,1}(W(1)) = \frac{1}{2}F(1)W(1) = 0, \\
 W(2) &= \frac{4}{2}\left(\frac{2}{4}H(0)W(2) + \frac{1}{2}H(1)W(1)\right) = 0, \\
 k = 2, H(2) &= \left(\frac{1}{2}\right)^2\left(\sum_{l=1}^k F(l)\hat{B}_{2,1}(W(1), W(2))\right) \\
 &= \frac{1}{4}(F(1)W(2) + F(2)W^2(1)) = -\frac{1}{2}, \\
 W(3) &= \frac{4}{3}\left(\frac{3}{8}H(0)W(3) + \frac{2}{4}H(1)W(2) + \frac{1}{2}H(2)W(1)\right) \\
 &= -\frac{4}{3}, \dots \text{ and so on.} \tag{10.61}
 \end{aligned}$$

Now, with the help of equation (10.4), the series solution is given by

$$w(v) = 2v - \frac{4}{3}v^3 + \dots, \tag{10.62}$$

which converges to the exact solution given by equation (10.57).

The approximate solution for $N = 12$ is compared with the analytical solution in Table (10.12), where N represents a number of terms considered. Table (10.13) lists the maximal absolute error of approximate results obtained by the present method for $N = 5, 10$, and 15 . Figure 10.5 depicts absolute errors for the numerical solutions for $N = 5, 10$, and 15 . From these results, it

TABLE 10.12: Comparison of numerical solution $w(v)$ with exact solution when $N = 12$ for Example 5

v	$w(v)$	$w_N(v)$	$R_N(v)$
0.1	0.1986693308	0.1986693308	7.2E-16
0.2	0.3894183423	0.3894183423	1.3E-16
0.3	0.5646424734	0.5646424734	2.8E-15
0.4	0.7173560909	0.7173560909	3.7E-13
0.5	0.8414709848	0.8414709846	1.2E-11
0.6	0.9320390860	0.9320390843	1.8E-09
0.7	0.9854497300	0.9854497174	1.2E-08
0.8	0.9995736030	0.9995735316	7.1E-08
0.9	0.9738476309	0.9738473016	3.3E-07
1.0	0.9092974268	0.9092961360	1.4E-02

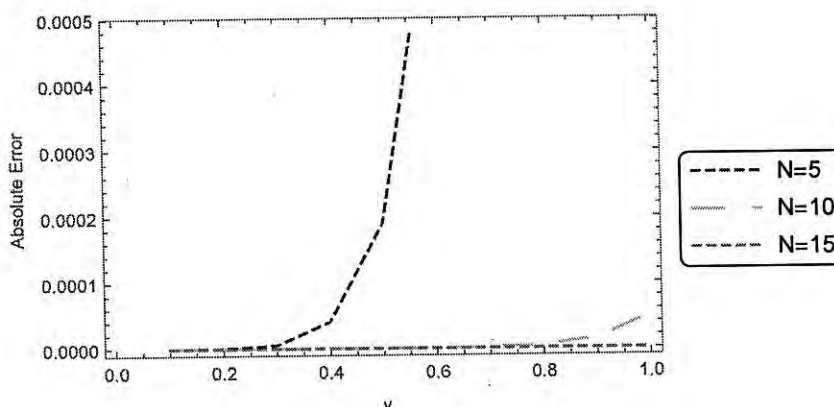
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TABLE 10.13: Maximum absolute errors for w of Example 5

N	$E_{N,\infty}$
5	2.4E-02
10	5.0E-05
15	3.6E-10

**FIGURE 10.5:** Absolute errors for $N = 5, 10,$ and 15 of Example 5.

is clear that absolute errors and maximal absolute errors all decline systematically with the increase in N .

Bibliography

- [1] Adak, D., Bairagi, N. and Haki, R. 2020. Chaos in delay-induced Leslie-Gower prey-predator-parasite model and its control through prey harvesting, *Nonlinear Analysis: Real World Appl.* 5: 102998–103020.
- [2] Ajello, W.G., Freedman, H.I. and Wu, J. 1992. A model of stage structured population growth with density depended time delay, *SIAM J. Appl. Math.* 52: 855–869.
- [3] Benhammouda, B. and Hector, V.L. 2016. A new multi step technique with differential transform method for analytical solution of some non-linear variable delay differential equations, *Springer Plus* 5: 1–17.
- [4] Bellour, A. and Bousselsal, M. 2014. Numerical solution of delay integro-differential equations by using Taylor collocation method, *Math. Methods Appl. Sci.* 37: 1491–1506.

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- [5] Blanco-Cocom, L., Estrella, A.G. and Avila-Vales, E. 2013. Solving delay differential systems with history functions by the Adomian decomposition method, *Appl. Math. Comput.* 218: 5994–6011.
- [6] Buhmann, M.D. and Iserles, A. 1993. Stability of the discretized pantograph differential equation, *Math. Comput.* 60: 575–589.
- [7] Chen, C.K. and Ho, S.H. 1999. Solving partial differential equations by two-dimensional differential transform method, *Appl. Math. Comput.* 106: 171–179.
- [8] Chen, X. and Wang, L. 2010. The variational iteration method for solving a neutral functional-differential equation with proportional delays, *Comput. Math. Appl.* 59: 2696–2702.
- [9] Cherepennikov, V. 2013. Numerical analytical method of studying some linear functional differential equations, *Numer. Anal. Appl.* 6: 236–246.
- [10] Comtet, L. 1974. *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Springer Science & Business Media.
- [11] Dugard, L. and Ei, V. 1997. *Stability and Control of Time-delay Systems*, Verlag Berlin Heidelberg.
- [12] Duarte, J., Januario, C. and Martins, N. 2016. Analytical solutions of an economic model by the homotopy analysis method, *App. Math Sci.*, 10:49: 2483–2490.
- [13] Ayaz, F. 2004. Solution of the systems of differential equations by differential transform method, *Appl. Math. Comput.* 147: 547–567.
- [14] Fox, L., Mayers, D.F., Ockendon, J.A. and Tayler, A.B. 1971. On a functional differential equation, *J. Inst. Math. Appl.* 8: 271–307.
- [15] Herrera, A.R. 2013. Chaos in delay differential equations with applications in population dynamics, *Discrete Contin. Dyn. Syst.* 33: 1633–1644.
- [16] Kuang, Y. 1993. *Delay Differential Equations with Applications in Population Dynamics*, Academic Press.
- [17] Methi, G. and Kumar, A. 2019. Numerical Solution of Linear and Higher-order Delay Differential Equations using the Coded Differential Transform Method, *Comput. Res. Model.*, 11:6: 1091–1099.
- [18] Ockendon, J.R. and Tayler, A.B. 1971. The dynamics of a current collection system for an electric locomotive, *Proc. Roy. Soc. London Ser. A*, 6: 447–468.
- [19] Pukhov, G.E. 1982. Differential transforms and circuit theory, *Circuit Theory Appl.* 10: 265–276.

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- [20] Ravi Kanth, A.S.V. and Aruna, K. 2008. Solution of singular two-point boundary value problems using differential transformation method, *Physics Letters A*, 372: 4671–4673.
- [21] Rebenda, J. and Šmarda, Z. 2017. A differential transformation approach for solving functional differential equations with multiple delays, *Commun. Nonlinear Sci. Numer. Simul.* 48: 246–257.
- [22] Rebenda, J. 2018. An application of Bell polynomials in numerical solving of nonlinear differential equations, *Aplimat Proceedings* 2018: 1–10.
- [23] Rebenda, J. and Šmarda, Z. 2019. Numerical algorithm for nonlinear delayed differential systems of n th order, *Adv. Diff. Equ.* 26: 1–13.
- [24] Rebenda, J. and Pátíková, Z. 2020. Differential Transform Algorithm for Functional Differential Equations with Time-Dependent Delays, *Complexity* 2020: 1–12.
- [25] Sezer, M., Yalcinbas, S. and Sahin, N. 2008. Approximate solution of multi-pantograph equation with variable coefficients, *J. Comput. Appl. Math.* 214: 406–416.
- [26] Shakeri, F. and Dehghan, M. 2008. Solution of delay differential equations via a homotopy perturbation method, *Math. Comput. Model.* 48: 486–498.
- [27] Warne, P., Warne, D., Sochacki, J., Parker G. and Carothers, D. 2006. Explicit A-Priori error bounds and adaptive error control for approximation of nonlinear initial value differential systems, *Comput. Math. Appl.* 52:12: 1695–1710.
- [28] Widatalla, S and Koroma, M. 2012. Approximation algorithm for a system of pantograph equations, *J. Appl. Math.* 2012: 1–9.
- [29] Yu, Z.H. 2008. Variational iteration method for solving the multi-pantograph delay equation, *Phys. Lett. A*, 372: 6475–6479.
- [30] Zhao, J.J., Xu, Y., Wang, H.X. and Liu, M.Z. 2006. Stability of a class of Runge-Kutta methods for a family of pantograph equations of neutral type, *Appl. Math. Comput.* 181:2: 1170–1181.

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