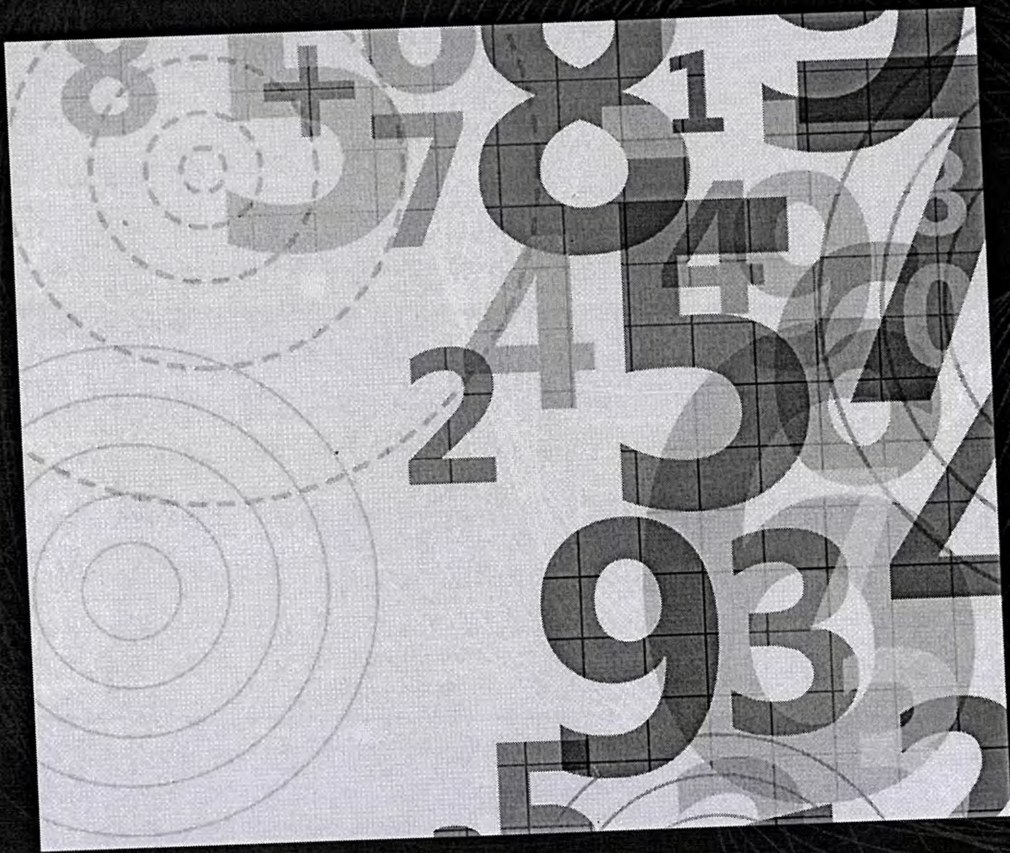
SEQUENCE SPACE THEORY WITH APPLICATIONS



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Sequence Space Theory with Applications

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Chapter 11

Carathéodory Theory of Dynamic Equations on Time Scales

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11.1 Introduction and Preliminaries

The time scales calculus and the theory of dynamic equations on time scales appears in the literature in the past three decades. It provides a unified framework in which one can study both continuous and discrete dynamical process simultaneously. Hence, it is realistic to model a hybrid phenomenon involving continuous and discrete data simultaneously, using dynamic equations on time scales. Thus, this topic has gained significant attention from many researchers. However, in the literature, little study has been done for dynamic equations on time scales involving Carathéodory function. In this chapter, we will investigate Carathéodory solutions of dynamic initial value problem of the type

$$\begin{cases} x^{\Delta} = f(t,x), & t \in [a,b]_{\mathbb{T}}^{\kappa}; \\ x(a) = A, \end{cases} \tag{11.1}$$

where $A \in \mathbb{R}$ and $f : [a,b]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is a Δ-Carathéodory function, and x^{Δ} is the delta derivative $x : [a,b)_{\mathbb{T}} \to \mathbb{R}$.

In the Carathéodory theory, we study the solutions of dynamic equations under weak conditions that the right-hand side of the dynamic equations is not necessarily continuous. This allows us to broaden the class of dynamic

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equations beyond those with continuous right-hand sides. We assume that the reader is familiar with the classical theory of Carathéodory. In this section, several concepts and results of the time scale calculus are briefly exposed that provide a framework for the study of dynamic equations in the sense of Carathéodory. The reader can find information on time scales calculus, for examples, in [2,3].

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$, with the subspace topology inherited from the standard topology of $\mathbb{R}$. For $a, b \in \mathbb{T}$ with $a < b$, the time scale interval $[a,b]_{\mathbb{T}}$ is defined as $[a,b] \cap \mathbb{T} := \{t \in \mathbb{T} : a \leq t \leq b\}$. For $t \in \mathbb{T}$, we define two operators, $\sigma : \mathbb{T} \to \mathbb{T}$ as $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, called *the forward jump operator* and $\rho : \mathbb{T} \to \mathbb{T}$ as $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ called *the backward jump operator*. With the help of these two jump operators, we classify the points in a time scale $\mathbb{T}$ in the following way: A point $t \in \mathbb{T}$ is *right-scattered* if $\sigma(t) > t$; while it is *left-scattered* if $\rho(t) < t$. A point $t \in \mathbb{T}$ is *right-dense* if $\sigma(t) = t$; while it is *left-dense* if $\rho(t) = t$. A point $t \in \mathbb{T}$ is *dense* if $\rho(t) = t = \sigma(t)$; while it is *isolated* if $\rho(t) < t < \sigma(t)$. The *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$. The metric on $\mathbb{T}$ is defined by $d(r,s) := |\mu(r,s)|$, where $\mu(t,\tau) = \mu(t)$ for a fixed $\tau \in \mathbb{T}$. This metric d generates the order topology on time scale $\mathbb{T}$.

Below we list some standard notations that are being used in this chapter. The set of continuous functions from $[a,b]_{\mathbb{T}}$ to $\mathbb{R}$ is denoted by $\mathrm{C}([a,b]_{\mathbb{T}}, \mathbb{R})$. The set of absolutely continuous functions from $[a,b]_{\mathbb{T}}$ to $\mathbb{R}$ is denoted by $\mathrm{AC}([a,b]_{\mathbb{T}}, \mathbb{R})$. The set of Δ-measurable functions from $[a,b]_{\mathbb{T}}$ to $\mathbb{R}$ is denoted by $\mathrm{M}([a,b]_{\mathbb{T}}, \mathbb{R})$. The set of bounded Δ-measurable functions from $[a,b]_{\mathbb{T}}$ to $\mathbb{R}$ is denoted by $\mathrm{BM}([a,b]_{\mathbb{T}}, \mathbb{R})$. The set of Lebesgue Δ-integrable functions from $[a,b]_{\mathbb{T}}$ to $\mathbb{R}_+$ is denoted by $\mathrm{L}_1([a,b]_{\mathbb{T}}, \mathbb{R}_+)$. We note that

$$\mathrm{AC}([a,b]_{\mathbb{T}}, \mathbb{R}) \subset \mathrm{C}([a,b]_{\mathbb{T}}, \mathbb{R}) \subset \mathrm{BM}([a,b]_{\mathbb{T}}, \mathbb{R}) \subset \mathrm{M}([a,b]_{\mathbb{T}}, \mathbb{R}).$$

For $x : [a,b]_{\mathbb{T} \to \mathbb{R}}$, we define *sup-norm* on the set $\mathrm{BM}([a,b]_{\mathbb{T}}, \mathbb{R})$ as $\|x\|_0 := \sup\limits_{t \in [a,b]_{\mathbb{T}}} |x(t)|$. Also, we define the *generalized Bielecki's norm*, called *TZ-norm* on the space $\mathrm{BM}([a,b]_{\mathbb{T}}, \mathbb{R})$ as

$$\|x\|_\beta := \sup_{t \in [a,b]_{\mathbb{T}}} \frac{|x(t)|}{e_\beta(t,a)}.$$

We note that the sup-norm $\|\cdot\|_0$ and the TZ-norm $\|\cdot\|_\beta$ are equivalent. Since $(\mathrm{BM}([a,b]_{\mathbb{T}}, \mathbb{R}), \|\cdot\|_0)$ is a Banach space, it follows that $(\mathrm{BM}([a,b]_{\mathbb{T}}, \mathbb{R}), \|\cdot\|_\beta)$ is also a Banach space.

Definition 11.1.1. *(See [2, Definition 1.58]) A function $x : [a,b)_{\mathbb{T}} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at every right-dense points in $[a,b)_{\mathbb{T}}$ and its left sided limits exist at left dense points in $[a,b)_{\mathbb{T}}$. The notation $\mathrm{C}_{\mathrm{rd}}([a,b)_{\mathbb{T}}, \mathbb{R})$ is used to denote the set of all rd-continuous functions $x : [a,b)_{\mathbb{T}} \to \mathbb{R}$.*

Definition 11.1.2. *(See [2, Definition 2.45]) A function $\beta : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is said to be positively regressive if $1 + \mu(t)\beta(t) > 0$ for all $t \in [a,b]_{\mathbb{T}}$. The notation $\mathcal{R}^+$ is used to denote the set of all positively regressive rd-continuous functions $\beta : [a,b]_{\mathbb{T}} \to \mathbb{R}$.*

Definition 11.1.3. *(See [2, Definition 2.30]) For $\beta \in \mathcal{R}^+$, the exponential function $e_\beta(\cdot, a)$ is defined as*

$$e_\beta(t,a) := \begin{cases} \exp\left(\int_a^t \beta(s)ds\right) & \text{if } \mu(t) = 0; \\ \exp\left(\int_a^t \frac{Log(1+\mu(s)\beta(s))}{\mu(s)} \Delta s\right) & \text{if } \mu(t) > 0, \end{cases}$$

for all $t \in [a,b]_{\mathbb{T}}$, where Log *is the principal logarithm function.*
Since $\beta \in \mathcal{R}^+$, we note that $e_\beta(t,a) > 0$ for all $t \in [a,b]_{\mathbb{T}}$ and $e_\beta(a,a) = 1$.

Definition 11.1.4. *[4] A function $x : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is said to be absolutely continuous if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\sum_{i=1}^{n} |x(d_i) - x(c_i)| < \varepsilon,$$

whenever $\{[c_i, d_i]_{\mathbb{T}} : 1 \le i \le n\}$ is a finite family of pairwise disjoint subintervals of $[a,b]_{\mathbb{T}}$ obeying

$$\sum_{i=1}^{n} (d_i - c_i) < \delta.$$

A function $x : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is called an arc if it is absolutely continuous.

Theorem 11.1.5. *(Fundamental theorem of calculus for vector valued functions, [4, Theorem 4.1]) A function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is an arc if and only if the following hold.*

(i) *for Δ-a.e. $t \in [a,b)_{\mathbb{T}}$, the function f is Δ-differentiable and $f^\Delta \in L_1([a,b)_{\mathbb{T}}, \mathbb{R})$;*

(ii) *for each $t \in [a,b]_{\mathbb{T}}$, we have*

$$f(t) = f(a) + \int_a^t f^\Delta(s)\Delta s.$$

Theorem 11.1.6. *(See [8, Proposition 2.4]) Let $\{x_i\}$ be a sequence of arcs $x_i : \mathbb{T} \to \mathbb{R}$ and let $\phi : \mathbb{T} \to [0, +\infty)$ be in $L_1([a,b)_{\mathbb{T}}, \mathbb{R}_+)$ such that for each $i \in \mathbb{N}$, we get*

$$\|x_i^\Delta(s)\| \le \phi(s) \quad \Delta\text{-a.e. } s \in [a,b)_{\mathbb{T}}. \tag{11.2}$$

If $\{x_i(a)\}$ is a bounded sequence, then there exists a subsequence $\{x_{i_k}\} \subset \{x_i\}$ and an arc $x : \mathbb{T} \to \mathbb{R}$ such that $x_{i_k} \rightrightarrows x$. Furthermore, if $c, d \in \mathbb{T}$ with $c < d$, then

$$\lim_{k\to\infty} \int_c^d x_{i_k}^\Delta(s)\Delta s = \int_c^d x^\Delta(s)\Delta s.$$

Remark 11.1.7. *The above theorem remains valid if the condition given in* (11.2) *is replaced by*

$$\left|\int_c^d x_i^{\Delta}(s)\Delta s\right| \le \int_c^d \phi(s)\Delta s$$

for each $c, d \in \mathbb{T}$ *with* $c < d$.

Theorem 11.1.8. *(Measure continuity [8, Proposition 1.11]) Let* $f : [a,b]_{\mathbb{T}} \to [0, +\infty)$ *be a Lebesgue* Δ*-integrable function. For every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that, if* A *is a* Δ*-measurable subset of* $[a,b]_{\mathbb{T}}$ *with* $\mu_{\Delta}(A) < \delta$*, then* $\displaystyle\int_A f(s)\Delta s < \varepsilon$.

Theorem 11.1.9. *(See [5, Proposition 4.1]) Let* $f : \mathbb{T} \to \mathbb{R}$ *be given function and* $\tilde{f} : [a,b] \to \mathbb{R}$ *be its extension defined as*

$$\tilde{f}(t) := \begin{cases} f(t) & \text{if } t \in \mathbb{T}, \\ f(t_i) & \text{if } t \in (t_i, \sigma(t_i)), \text{ for some } i \in I \subset \mathbb{N}. \end{cases} \tag{11.3}$$

Then, f *is* Δ*-measurable if and only if* $\tilde{f}$ *is Lebesgue measurable.*

Lemma 11.1.10. *(See [5, Lemma 3.2]) Let* $E \subset [a,b)_{\mathbb{T}}$ *be such that* E *does not contain any right-scattered points. Then* $m^*(E) = \mu^*(E)$.

Theorem 11.1.11. *(See [5, Theorem 5.1]) Let* $E \subset \mathbb{T}$ *be a* Δ*-measurable set such that* $b \notin E$*. Define the set* $\tilde{E}$ *as*

$$\tilde{E} := E \cup \bigcup_{i \in I_E} (t_i, \sigma(t_i)),$$

where $I_E := \{i \in I : t_i \in E \cap R\}$ *with* $I \subset \mathbb{N}$ *and* $R := \{t \in \mathbb{T} : t < \sigma(t)\}$*. Let* $f : \mathbb{T} \to \mathbb{R}$ *be a* Δ*-measurable function and* $\tilde{f} : [a,b] \to \mathbb{R}$ *be its extension defined in* (11.3). *Then,* $f \in \mathrm{L}_1(E, \mathbb{R})$ *if and only if* $\tilde{f} \in \mathrm{L}_1(\tilde{E}, \mathbb{R})$*. Further*

$$\int_E f(s)\Delta s = \int_{\tilde{E}} \tilde{f}(s)ds.$$

Definition 11.1.12. *(See [6, Definition 2.30]) A function* $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ *is said to be* Δ*-Carathéodory function if it satisfies the following conditions.*

(C-i) *The map* $t \mapsto f(t,x)$ *is* Δ*-measurable for every* $x \in \mathbb{R}$.

(C-ii) *The map* $x \mapsto f(t,x)$ *is continuous* Δ*-a.e.* $t \in [a,b]_{\mathbb{T}}$.

(C-iii) *For every* $r > 0$*, there exists* $h_r \in \mathrm{L}_1([a,b]_{\mathbb{T}}, \mathbb{R}_+)$ *such that*

$$|f(t,x)| \le h_r(t)$$

for Δ*-a.e.* $t \in [a,b]_{\mathbb{T}}$ *and* $|x| < r + |x_0|$.

Definition 11.1.13. *An arc* $x : [a,b)_{\mathbb{T}} \to \mathbb{R}$ *is said to be a solution of dynamic problem* (11.1)*, if it satisfy* (11.1)*. A solution of* (11.1) *is* Δ*-Carathéodory solution if the function* f *in* (11.1) *is* Δ*-Carathéodory function.*

11.2 Carathéodory Solutions

We shall deal with the Carathéodory solutions of dynamic problem (11.1). We start with the following useful lemma.

Lemma 11.2.1. *Let $x : \mathbb{T} \to \mathbb{R}$ be an arc and $f \in L_1([a,b)_{\mathbb{T}}, \mathbb{R})$ be such that*

$$x(t) = \int_c^t f(s)\Delta s,$$

for each $t \in [c,d]_{\mathbb{T}}$, where $c, d \in \mathbb{T}$ with $c < d$. Then

$$x^{\Delta}(t) = f(t) \quad \Delta\text{-a.e. } t \in [c,d)_{\mathbb{T}}.$$

Proof. Consider all those points $t \in [c,d)_{\mathbb{T}}$ where $x^{\Delta}(t)$ exists. If $\sigma(t) > t$, then by definition

$$x^{\Delta}(t) = \frac{\int_c^{\sigma(t)} f(s)\Delta s - \int_c^t f(s)\Delta s}{\mu(t)} = \frac{\int_t^{\sigma(t)} f(s)\Delta s}{\mu(t)} = f(t).$$

If $\sigma(t) = t$ and t is a Lebesgue point of $\tilde{f}$, then there is a sequence $\{t_i\}_{i\in\mathrm{N}}$ in $[c,d)_{\mathbb{T}}$ such that $t_i \to t$. Now, using Theorem 11.1.11, we can write

$$x^{\Delta}(t) = \lim_{i\to\infty} \frac{x(t_i) - x(t)}{t_i - t} = \lim_{i\to\infty} \frac{\int_t^{t_i} f(s)\Delta s}{t_i - t} = \lim_{i\to\infty} \frac{\int_t^{t_i} \tilde{f}(s)ds}{t_i - t} = \tilde{f}(t) = f(t).$$

Thus, if $D := \{t \in [c,d)_{\mathbb{T}} : x^{\Delta}(t) \neq f(t)\}$, then $D \subset R \cap E$, where $R := \{t \in [c,d)_{\mathbb{T}} : \sigma(t) = t\}$ and E is the set of points $t \in [c,d)_{\mathbb{T}}$ such that t is not a Lebesgue point of $\tilde{f}$. From Lemma 11.1.10, we find that

$$m^*(D) \leq m^*(R \cap E) = \mu^*(R \cap E) \leq \mu^*(E) = 0.$$

Hence, $x^{\Delta}(t) = f(t)$ Δ-a.e. $t \in [c,d)_{\mathbb{T}}$. $\square$

Lemma 11.2.2. *Let $f : [a,b]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ be an integrable function.*

1. *If an arc $x : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is a solution of dynamic problem* (11.1), *then it follows that*

$$x(t) = A + \int_a^t f(s, x(s))\Delta s \quad \forall\, t \in [a,b]_{\mathbb{T}}. \tag{11.4}$$

2. *Every arc $x : [a,b]_{\mathbb{T}} \to \mathbb{R}$ obeying* (11.4) *is a solution of dynamic problem* (11.1).

Theorem 11.2.3. *Let $f : [a,b]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ be a Δ-Carathéodory function. Then there exists $b_1 \in \mathbb{T} \setminus \{a\}$ such that the dynamic problem* (11.1) *defined on $[a,b_1)_{\mathbb{T}}$ has a Carathéodory solution.*

Proof. First, we assume that a is right-scattered point, $\sigma(a) > a$. Then take $b_1 = \sigma(a)$. Define an arc $x : [a,b_1]_{\mathbb{T}} \to \mathbb{R}$ by

$$x(t) := \begin{cases} A, & \text{if } t = a, \\ \mu(a) f(a, x(a)) + x(a), & \text{if } t = b_1. \end{cases}$$

Then, by definition

$$x^{\Delta}(a) = \frac{x(\sigma(a)) - x(a)}{\sigma(a) - a} = \frac{x(\sigma(a)) - x(a)}{\mu(a)} = \frac{\mu(a) f(a, x(a))}{\mu(a)} = f(a, x(a)).$$

Thus, $x^{\Delta}(t) = f(t, x(t))$ for $t \in [a, b_1)_{\mathbb{T}}$. Hence, x is a Carathéodory solution to dynamic problem (11.1). Next, we assume that a is right-dense point. That is, $\sigma(a) = a$. Let $r > 0$ be arbitrarily fixed. Then, by Theorem 11.1.8, there exists $b_1 \in \mathbb{T} \setminus \{a\}$ such that

$$\int_a^{b_1} h_r(s) \Delta s \leq r.$$

We define the set $\mathcal{A}$ to be the set of all arcs $x : \mathbb{T} \to \mathbb{R}$ satisfying $|x(t) - A| \leq r$ for all $t \in [a, b_1]_{\mathbb{T}}$ and

$$\left| \int_c^d x^{\Delta}(s) \Delta s \right| \leq \int_c^d h_r(s) \Delta s$$

for each $c, d \in \mathbb{T}$ such that $c < d$. Using Theorem 11.1.6, we conclude that the set $\mathcal{A}$ is a compact set. Also, $\mathcal{A}$ is a convex set. Now, for each $x \in \mathcal{A}$, we define the function $y_x : \mathbb{T} \to \mathbb{R}$ by

$$y_x(t) := \begin{cases} x(t), & \text{if } t \in [a, b_1]_{\mathbb{T}}, \\ x(b_1), & \text{if } t \in \mathbb{T} \setminus [a, b_1]_{\mathbb{T}}. \end{cases}$$

Note that the function y_x is a continuous function. Therefore, the function $f(s, y_x(s))$ is Δ-measurable. Define the mapping $F : \mathcal{A} \to \mathrm{C}(\mathbb{T}, \mathbb{R})$ by

$$F[x](t) := A + \int_a^t f(s, y_x(s)) \Delta s \tag{11.5}$$

for all $t \in \mathbb{T}$ and all $x \in \mathcal{A}$. Then, by Theorem 11.1.8, Fx is an arc, for arbitrarily fixed $x \in \mathcal{A}$. From Lemma 11.2.1, it follows that

$$(F[x](t))^{\Delta} = f(t, y_x(t)) \quad \Delta\text{-a.e. } t \in [a, b)_{\mathbb{T}}.$$

Therefore, for each $c, d \in \mathbb{T}$ such that $c < d$, we have

$$\left|\int_c^d (F[x](s))^{\Delta} \Delta s\right| = \left|\int_c^d f(s, y_x(s))\Delta s\right| \leq \int_c^d h_r(s)\Delta s.$$

Now, for each $t \in [a, b_1]_{\mathbb{T}}$, we deduce that

$$\begin{aligned}|F[x](t) - A| = \left|\int_a^t f(s, y_x(s))\Delta s\right| \leq \int_a^t |f(s, y_x(s))|\Delta s &\leq \int_a^t h_r(s)\Delta s \\ &\leq \int_a^{b_1} h_r(s)\Delta s \\ &\leq r.\end{aligned}$$

Hence, we obtain $F(\mathcal{A}) \subset \mathcal{A}$. Let $\bar{x} \in \mathcal{A}$ be arbitrary. Then, there is a sequence $\{x_i\}_{i\in\mathbb{N}}$ in $\mathcal{A}$ such that $\|x_i - \bar{x}\|_{\infty} \to 0$ as $i \to \infty$. Thus, for each $s \in \mathbb{T}$,

$$\lim_{i\to\infty} |f(s, y_{x_i}(s)) - f(s, y_{\bar{x}}(s))| = 0$$

and

$$|f(s, y_{x_i}(s)) - f(s, y_{\bar{x}}(s))| \leq 2h_r(s)$$

for all $i \in \mathbb{N}$. By Lebesgue dominated convergence theorem, we can write

$$\lim_{i\to\infty} \int_a^b |f(s, y_{x_i}(s)) - f(s, y_{\bar{x}}(s))| \Delta s = 0$$

and then $\lim_{i\to\infty} |Fx_i - F\bar{x}| = 0$. Hence, the mapping F is continuous at $\bar{x}$. Therefore, by Schaefer's fixed point theorem [12], there exists $x^* \in \mathcal{A}$ such that $F(x^*) = x^*$. From Lemma 11.2.1, we may conclude that $x^{*\Delta}(t) = f(t, x^*(t))$ Δ- a.e. $t \in [a, b_1)_{\mathbb{T}}$. Hence, x^* is a solution of dynamic problem (11.1). □

Theorem 11.2.4. *Let $f : [a, b]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ be a Δ-Carathéodory function. If there is a sequence $\{f_k\}$ of Δ-Carathéodory functions such that*

$$\int_a^t f_k(s, z_k(s))\Delta s \to \int_a^t f(s, x(s))\Delta s,$$

where $\{z_k\}$ is the sequence of arcs converges uniformly to x on $[a, b)_{\mathbb{T}}$, then there exists $b_1 \in [a, b)_{\mathbb{T}} \setminus \{a\}$ such that the dynamic problem (11.1) has a Δ-Carathéodory solution x on $[a, b_1)_{\mathbb{T}}$.

Proof. If a is right scattered point in $[a, b]_{\mathbb{T}}$, then take $b_1 = \sigma(a)$ and whence $[a, b_1)_{\mathbb{T}} = \{a\}$. Define the arc $x : [a, b_1]_{\mathbb{T}} \to \mathbb{R}$ by

$$x(t) := \begin{cases} A & \text{for } t = a, \\ f(a, x(a))\mu(a) + x(a) & \text{for } t = b_1. \end{cases}$$

Then

$$x(b_1) - x(a) = f(a, x(a))\mu(a) \text{ implies } \frac{x(b_1) - x(a)}{\mu(a)} = f\big(a, x(a)\big).$$

That is, $x^{\Delta}(t) = f(t, x(t))$ for $t \in [a, b_1)_{\mathbb{T}} = \{a\}$. Thus, x is a Δ-Carathéodory solution to dynamic problem (11.1). If $\sigma(a) = a$, then by Theorem 11.1.8, there exists $b_1 \in [a, b)_{\mathbb{T}} \setminus \{a\}$ such that

$$\int_a^{b_1} h_r(s)\Delta s < r \tag{11.6}$$

for arbitrarily fixed $r > 0$. Since $\{f_i\}$ is a sequence of Δ-Carathéodory functions, by Theorem 11.2.3, there is a sequence of $\{x_i\}$ of arcs $x_i : [a, b_1)_{\mathbb{T}} \to \mathbb{R}$, $b_1 \in [a, b)_{\mathbb{T}} \setminus \{a\}$ obeying (11.1). Therefore, each x_i is a Δ-Carathéodory solution of dynamic problem(11.1) on $[a, b_1)_{\mathbb{T}}$. By Lemma 11.2.2, for $t \in [a, b_1)_{\mathbb{T}}$ and for each $i \in \mathbb{N}$, we obtain

$$x_i(t) = A + \int_a^t f_i(s, x_i(s))\Delta s.$$

Then

$$|x_i(t)| \le |A| + \int_a^t |f_i(s, x_i(s))|\Delta s \le |A| + \int_a^{b_1} h_r(s)\Delta s$$

which, by (11.6), yields $|x_i(t)| \le |A| + r$ $(r > 0)$. Hence, $\{x_i\}$ is uniformly bounded on $[a, b_1)_{\mathbb{T}}$. Now, for $t_1, t_2 \in [a, b_1)_{\mathbb{T}}$, we have

$$\begin{aligned}
|x_i(t_1) - x_i(t_2)| &= \left|\int_a^{t_1} f_i(s, x_i(s))\Delta s - \int_a^{t_2} f_i(s, x_i(s))\Delta s\right| \\
&\le \int_{t_1}^{t_2} |f_i(s, x_i(s))|\Delta s \\
&\le \int_{t_1}^{t_2} h_r(s)\Delta s.
\end{aligned}$$

But, from Theorem 11.1.8, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $t_1, t_2 \in [a, b_1)_{\mathbb{T}}$ and $|t_1 - t_2| \le \delta$ imply

$$\int_{t_1}^{t_2} h_r(s)\Delta s < \varepsilon.$$

Consequently, $|x_i(t_1) - x_i(t_2)| < \varepsilon$. Therefore, $\{x_i\}$ is equicontinuous on $[a, b_1)_{\mathbb{T}}$. By the Arzelà–Ascoli theorem [1], there is a subsequence $\{y_j\}$ of $\{x_i\}$ which converges uniformly on $[a, b_1)_{\mathbb{T}}$ to an arc $x : [a, b_1)_{\mathbb{T}} \to \mathbb{R}$. We show that this x is solution of dynamic problem (11.1). Since $x_i(a) = A$, $x(a) = \lim_{j\to\infty} y_j(a) = A$. Let $t \in [a, b_1)_{\mathbb{T}}$ be fixed. Then, by hypothesis,

$$\lim_{k\to\infty} \int_a^t f_k(s, z_k(s))\Delta s = \int_a^t f(s, x(s))\Delta s,$$

where $\{z_k\}$ is a subsequence of $\{y_j\}$. Since $z_k(t) = A + \int_a^t f_k(s, z_k(s))\Delta s$ and $\lim_{k\to\infty} z_k(t) = x(t)$, it follows that

$$\lim_{k\to\infty}\left(A + \int_a^t f_k(s, z_k(s))\Delta s\right) = x(t),$$

and we obtain $A + \int_a^t f(s, x(s))\Delta s = x(t)$ which, by Lemma 11.2.2, proves that x is a solution of dynamic problem (11.1). □

Theorem 11.2.5. *Let $f : [a,b]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ be a Δ-Carathéodory function. If there exists a positively regressive and rd-continuous function $\beta : [a,b]_{\mathbb{T}} \to \mathbb{R}$ such that $|f(t,u) - f(t,v)| \le \beta(t)|u - v|$ for Δ-a.e. $t \in [a,b]_{\mathbb{T}}$ and for all $u, v \in \mathbb{R}$, then the dynamic problem* (11.1) *has a unique solution x such that $\|x\|_\beta \le |A| + M\|h_r\|_\beta$ for some positive constant M.*

Proof. Define the TZ-norm $\|\cdot\|_\beta$ on the space $BM([a,b]_{\mathbb{T}}, \mathbb{R})$ as

$$\|x\|_\beta = \sup_{t\in[a,b]_{\mathbb{T}}} \frac{|x(t)|}{e_\beta(t,a)},$$

where $\beta : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is regressive, rd-continuous function. Then $|x(t)| \le \|x\|_\beta\, e_\beta(t,a)$ for all $t \in [a,b]_{\mathbb{T}}$. Let $F : \mathrm{BM}([a,b]_{\mathbb{T}}, \mathbb{R}) \to \mathrm{BM}([a,b]_{\mathbb{T}}, \mathbb{R})$ be defined by

$$F[x](t) = A + \int_a^t f(s, x(s))\Delta s, \quad t \in [a,b]_{\mathbb{T}}.$$

Let $x_1, x_2 \in \mathrm{BM}([a,b]_{\mathbb{T}}, \mathbb{R})$. Then, for all $t \in [a,b]_{\mathbb{T}}$,

$$F[x_1](t) = A + \int_a^t f(s, x_1(s))\Delta s \text{ and } F[x_2](t) = A + \int_a^t f(s, x_2(s))\Delta s.$$

Therefore,

$$\begin{aligned} |F[x_1](t) - F[x_2](t)| &= \left|\int_a^t [f(s, x_1(s)) - f(s, x_2(s))]\,\Delta s\right| \\ &\le \int_a^t \beta(s)|x_1(s) - x_2(s)|\Delta s. \end{aligned}$$

Then

$$\begin{aligned} \frac{|F[x_1](t) - F[x_2](t)|}{e_\beta(t,a)} &\le \frac{1}{e_\beta(t,a)}\int_a^t \beta(s)\,|x_1(s) - x_2(s)|\,\Delta s \\ &\le \frac{1}{e_\beta(t,a)}\int_a^t \beta(s)e_\beta(s,a)\|x_1 - x_2\|_\beta \Delta s \\ &= \frac{1}{e_\beta(t,a)}\int_a^t e_\beta^\Delta(s,a)\|x_1 - x_2\|_\beta \Delta s \\ &= \left[1 - \frac{1}{e_\beta(t,a)}\right]\|x_1 - x_2\|_\beta. \end{aligned}$$

This yields

$$\sup_{t\in[a,b]_{\mathbb{T}}} \frac{|F[x_1](t) - F[x_2](t)|}{e_\beta(t,a)} \le \sup_{t\in[a,b]_{\mathbb{T}}} \left[1 - \frac{1}{e_\beta(t,a)}\right] \|x_1 - x_2\|_\beta$$
$$= \left[1 - \frac{1}{e_\beta(b,a)}\right] \|x_1 - x_2\|_\beta = \alpha\|x_1 - x_2\|_\beta$$

where $\alpha = 1 - \dfrac{1}{e_\beta(b,a)} < 1$. Hence,

$$\|Fx_1 - Fx_2\|_\beta \le \alpha\|x_1 - x_2\|_\beta \quad \text{and} \quad 0 < \alpha < 1 \ (b \ne a).$$

The Banach's fixed point theorem [12] assures that the function F has a unique fixed point in $\mathrm{BM}([a,b]_{\mathbb{T}}, \mathbb{R})$. This yields that the dynamic problem (11.1) has a unique solution. Now, for $t < b$, dividing throughout by $e_\beta(t,a)$ to (11.4), we get

$$\frac{|x(t)|}{e_\beta(t,a)} \le \frac{|A|}{e_\beta(t,a)} + \frac{1}{e_\beta(t,a)} \int_a^t |f(s, x(s))|\,\Delta s$$

and then

$$\begin{aligned}
\frac{|x(t)|}{e_\beta(t,a)} &\le \frac{|A|}{e_\beta(t,a)} + \frac{1}{e_\beta(t,a)} \int_a^t h_r(s)\Delta s \\
&\le \frac{|A|}{e_\beta(t,a)} + \frac{1}{e_\beta(t,a)}\|h_r\|_\beta \int_a^t e_\beta(s,a)\Delta s \\
&\le \frac{|A|}{e_\beta(t,a)} + \frac{1}{e_\beta(a,a)}\|h_r\|_\beta \int_a^t e_\beta(s,a)\Delta s \\
&\le \frac{|A|}{e_\beta(t,a)} + \|h_r\|_\beta \int_a^t e_\beta(s,a)\Delta s \\
&\le \frac{|A|}{e_\beta(t,a)} + \|h_r\|_\beta\,(b-a)e_\beta(b,a).
\end{aligned}$$

That is,

$$\frac{|x(t)|}{e_\beta(t,a)} \le \frac{|A|}{e_\beta(t,a)} + \|h_r\|_\beta\,(b-a)e_\beta(b,a) \le |A| + M\|h_r\|_\beta,$$

where $M := (b-a)e_\beta(b,a)$. Therefore $\|x\|_\beta \le |A| + M\|h_r\|_\beta$ which shows that the solution x is bounded with respect to $\|\cdot\|_\beta$. □

Theorem 11.2.6. *If $f : [a,b]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that there exist a constant $L > 0$ and a function $c \in L_1([a,b]_{\mathbb{T}}, \mathbb{R}_+)$ satisfying*

$$|f(t,x)| \le L|x| + c(t)$$

for Δ-a.e. $t \in [a,b)_{\mathbb{T}}$ and for all $x \in \mathbb{R}$, then the dynamic problem (11.1) *has at least one continuous solution.*

Proof. Define the operator $F : C([a,b]_{\mathbb{T}}, \mathbb{R}) \to C([a,b]_{\mathbb{T}}, \mathbb{R})$ by

$$F[x](t) := A + \int_a^t f\big(s, x(s)\big)\Delta s, \quad t \in [a,b]_{\mathbb{T}}. \tag{11.7}$$

We use Schaefer's fixed point theorem [12]. In view of Lemma 11.2.2, the fixed points of F will be solutions to the dynamic problem (11.1). We first show that F is continuous. For this, take a sequence $\{u_k\}$ in $C([a,b]_{\mathbb{T}}, \mathbb{R})$ such that $\|u_k - u\|_\beta \to 0$. Thence $\|u_k - u\|_0 \to 0$. Now, For $t \in [a,b]_{\mathbb{T}}$, we have

$$\begin{aligned} |F[u_k](t) - F[u](t)| &= \left| \int_a^t f(s, u_k(s))\Delta s - \int_a^t f(s, u(s))\Delta s \right| \\ &\le \int_a^t |f(s, u_k(s)) - f(s, u(s))|\, \Delta s \\ &\le \int_a^b |f(s, u_k(s)) - f(s, u(s))|\, \Delta s \end{aligned}$$

and then

$$\|Fu_k - Fu\|_0 \le \int_a^b |f(s, u_k(s)) - f(s, u(s))|\, \Delta s.$$

Since

$$|f(t, u_k(t)) - f(t, u(t))| \to 0 \quad \Delta\text{-a.e. } t \in [a,b]_{\mathbb{T}},$$

we may conclude that $\|Fu_k - Fu\|_0 \to 0$, and then $\|Fu_k - Fu\|_\beta \to 0$. Thus, F is a continuous map from $C([a,b]_{\mathbb{T}}, \mathbb{R})$ to itself. Let Ω be a bounded subset of $C([a,b]_{\mathbb{T}}, \mathbb{R})$. So, there exists a constant $k > 0$ obeying $\|u\|_0 \le k$, for all $u \in \Omega$. For $u \in \Omega$, we have

$$\begin{aligned} |F[u](t)| &= \left| A + \int_a^t f(s, u(s))\Delta s \right| \\ &\le |A| + \int_a^b |f(s, u(s))|\Delta s \\ &\le |A| + \int_a^b \left(L|u(s)| + c(s)\right)\Delta s \\ &\le |A| + \int_a^b L\|u\|_0 \Delta s + \int_a^b c(s)\Delta s \\ &\le |A| + Lk(b-a) + \int_a^b c(s)\Delta s \end{aligned}$$

and then

$$\|Fu\|_0 \le |A| + Lk(b-a) + \int_a^b c(s)\Delta s.$$

Therefore, $F(\Omega)$ is uniformly bounded. Next, for $t_1, t_2 \in [a,b]_{\mathbb{T}}$ with $t_1 < t_2$, we have

$$\begin{aligned}|F[u](t_1) - F[u](t_2)| &\leq \int_{t_1}^{t_2} |f(s, u(s))|\,\Delta s \\ &\leq \int_{t_1}^{t_2} (L|u(s)| + c(s))\,\Delta s \\ &\leq \int_{t_1}^{t_2} L\|u\|_0\ \Delta s + \int_{t_1}^{t_2} c(s)\Delta s \\ &\leq Lk(t_2 - t_1) + \int_{t_1}^{t_2} c(s)\Delta s.\end{aligned}$$

Let $\varepsilon > 0$ be given. Then, by Theorem 11.1.8, there exists a $\delta > 0$ such that $t_1, t_2 \in [a, b]_{\mathbb{T}}$ and $|t_1 - t_2| < \delta$ imply $|F[u](t_1) - F[u](t_2)| < \varepsilon$ for all $u \in \Omega$. Hence, $F(\Omega)$ is equicontinuous set in $\mathrm{C}([a, b]_{\mathbb{T}}, \mathbb{R})$ and the Arzelà–Ascoli theorem [1] assures that the set $F(\Omega)$ is relatively compact. Therefore, the map $F : \mathrm{C}([a, b]_{\mathbb{T}}, \mathbb{R}) \to \mathrm{C}([a, b]_{\mathbb{T}}, \mathbb{R})$ is completely continuous. Now, consider the set $\Gamma \subset \mathrm{C}([a, b]_{\mathbb{T}}, \mathbb{R})$ defined by

$$\Gamma := \big\{x \in \mathrm{C}([a, b]_{\mathbb{T}}, \mathbb{R}) : x = \lambda F x \text{ for some } \lambda \in [0, 1]\big\}.$$

We note that

$$\int_a^t L e_L(s, a)\ \Delta s = e_L(t, a) - 1 \text{ and } \|A\|_L = \sup_{t\in[a,b]_{\mathbb{T}}} \frac{|A|}{e_L(t, a)} = |A|.$$

If $x \in \Gamma$, it follows that

$$x(t) = \lambda A + \lambda \int_a^t f\big(s, x(s)\big)\Delta s.$$

Then $\|x\|_L = \lambda\|Fx\|_L \leq \|Fx\|_L$ and so

$$\begin{aligned}\|x\|_L &\leq \sup_{t\in[a,b]_{\mathbb{T}}} \frac{\left|A + \int_a^t f\big(s, x(s)\big)\Delta s\right|}{e_L(t, a)} \\ &\leq |A| + \sup_{t\in[a,b]_{\mathbb{T}}} \frac{\int_a^t |f(s, x(s))|\,\Delta s}{e_L(t, a)} \\ &\leq |A| + \sup_{t\in[a,b]_{\mathbb{T}}} \frac{\int_a^t \{L|x(s)| + c(s)\}\,\Delta s}{e_L(t, a)} \\ &= |A| + \sup_{t\in[a,b]_{\mathbb{T}}} \frac{\int_a^t L\ e_L(s, a)\frac{|x(s)|}{e_L(s, a)}\Delta s + \int_a^t c(s)\Delta s}{e_L(t, a)}\end{aligned}$$

$$\leq |A| + \sup_{t\in[a,b]_{\mathbb{T}}} \frac{\int_a^t L\, e_L(s,a)\|x\|_L\, \Delta s + \int_a^t c(s)\Delta s}{e_L(t,a)}$$

$$= |A| + \sup_{t\in[a,b]_{\mathbb{T}}} \frac{\|x\|_L\big(e_L(t,a) - 1\big) + \int_a^t c(s)\Delta s}{e_L(t,a)}$$

$$= |A| + \sup_{t\in[a,b]_{\mathbb{T}}} \left\{ \|x\|_L \left(1 - \frac{1}{e_L(t,a)}\right) + \frac{\int_a^t c(s)\Delta s}{e_L(t,a)} \right\}$$

$$\leq |A| + \|x\|_L \left(1 - \frac{1}{e_L(b,a)}\right) + \frac{\int_a^b c(s)\Delta s}{e_L(a,a)}$$

$$= |A| + \|x\|_L - \frac{\|x\|_L}{e_L(b,a)} + \int_a^b c(s)\Delta s.$$

Hence,

$$\|x\|_L \leq |A|\; e_L(b,a) + e_L(b,a)\int_a^b c(s)\Delta s$$

and therefore Γ is bounded. As a consequence of Schaefer's fixed point theorem [12], we deduce that, F has a fixed point u in $\mathrm{C}([a,b]_{\mathbb{T}};\mathbb{R})$, equivalently, the dynamic problem (11.1) has a solution in $\mathrm{C}\big([a,b]_{\mathbb{T}},\mathbb{R}\big)$. □

11.3 Generalized Dynamic Equations

11.3.1 Henstock–Kurzweil Δ-integral

In this section, the solutions of Carathéodory dynamic problem (11.1) is discussed, based on the concept of Henstock–Kurzweil delta integral.

Definition 11.3.1. *[11] A function $x : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is said to be absolutely continuous in the restricted sense on $E \subset [a,b]_{\mathbb{T}}$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\sum_{i=1}^{n} \operatorname{osc}(x, [c_i, d_i]_{\mathbb{T}}) < \varepsilon,$$

whenever $\{[c_i,d_i]_{\mathbb{T}} : 1 \leq i \leq n\}$ is a finite family of pairwise disjoint subintervals of $[a,b]_{\mathbb{T}}$ with $c_i, d_i \in E$ satisfying

$$\sum_{i=1}^{n} \mu_{\Delta}([c_i,d_i]_{\mathbb{T}}) < \delta.$$

We write $x \in \Delta\text{-AC}_*$. *A function* x *is said to be generalized absolutely continuous in the restricted sense if* x *is continuous on* $E \subset \mathbb{T}$ *and* E *can be written as countable union of sets on each of which* x *is* $\Delta\text{-AC}_*$. *In this case, we write* $x \in \Delta\text{-ACG}_*$. *If* x *is uniformly continuous on* E *and* E *can be written as countable union of sets on each of which* x *is* $\Delta\text{-AC}_*$, *then we say that* x *is uniformly* $\Delta\text{-ACG}_*$ *and write* $x \in \Delta\text{-UACG}_*$.

Definition 11.3.2. *A pair of functions* (δ_L, δ_R) *is said to be a* Δ*-gauge for* $[a,b]_{\mathbb{T}}$ *if* $\delta_L, \delta_R : [a,b]_{\mathbb{T}} \to \mathbb{R}$ *be two functions such that* $\delta_L(t) > 0$ *for* $t \in (a,b]_{\mathbb{T}}$, $\delta_L(a) \geq 0$, $\delta_R(t) > 0$ *for* $t \in [a,b)_{\mathbb{T}}$, $\delta_R(b) \geq 0$ *and* $\delta_R(t) \geq \mu(t)$ *for all* $t \in [a,b)_{\mathbb{T}}$. *We write* $\delta = (\delta_L, \delta_R)$.

Definition 11.3.3. *A division of* $[a,b]_{\mathbb{T}}$ *defined by*

$$\mathcal{P} := \{a = t_0 \leq \xi_1 \leq t_1 \leq \xi_2 \leq t_2 \ldots t_{n-1} \leq \xi_n \leq t_n = b\}$$

with $t_i > t_{i-1}$ *for* $1 \leq i \leq n$ *and* t_i, $\xi_i \in \mathbb{T}$ *is called as a partition of* $[a,b]_{\mathbb{T}}$. *The point* ξ_i *is called tag point associated with the subinterval* $[t_{i-1}, t_i]_{\mathbb{T}}$ *of* $[a,b]_{\mathbb{T}}$. *We denote such partition by* $\mathcal{P} = \{\xi_i, [t_{i-1}, t_i]_{\mathbb{T}}\}_{i=1}^{n}$.

Definition 11.3.4. *A partition* $\mathcal{P}$ *of* $[a,b]_{\mathbb{T}}$ *is* δ*-fine if*

$$\xi_i - \delta_L(\xi_i) \leq t_{i-1} < t_i \leq \xi_i + \delta_R(\xi_i)$$

for $1 \leq i \leq n$, *where* $\delta := (\delta_L, \delta_R)$ *is a* Δ*-gauge for* $[a,b]_{\mathbb{T}}$.

Definition 11.3.5. *A function* $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ *is said to be Henstock–Kurzweil* Δ*-integrable on* $[a,b]_{\mathbb{T}}$ *provided there is a real number* I *such that for every* $\varepsilon > 0$, *there is a* Δ*-gauge* δ, *for* $[a,b]_{\mathbb{T}}$ *such that for all* δ*-fine partitions* $\mathcal{P} = \{\xi_i, [t_{i-1}, t_i]_{\mathbb{T}}\}_{i=1}^{n}$ *of* $[a,b]_{\mathbb{T}}$,

$$\left| I - \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) \right| < \varepsilon.$$

The number I *is called as Henstock–Kurzweil* Δ*-integral of* f *on* $[a,b]_{\mathbb{T}}$ *and we write*

$$I = (HK) \int_a^b f(t)\Delta t.$$

Using Henstock–Kurzweil Δ-integral, we generalize the Δ-Carathéodory dynamic equation (11.1) as follows. The dynamic initial value problem (11.1) is said to be *generalized* Δ*-Carathéodory dynamic problem* if the function $f : [a,b]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions.

(C1) The map $t \mapsto f(t,x)$ is Δ-measurable for every $x \in \mathbb{R}$.

(C2) The map $x \mapsto f(t,x)$ is continuous Δ-a.e. $t \in [a,b]_{\mathbb{T}}$.

(C3) For every real number $r > 0$, there exist two Henstock–Kurzweil Δ-integrable functions g_r and h_r on $[a,b]_{\mathbb{T}}$ such that $g_r(t) \leq f(t,x) \leq h_r(t)$ for Δ-a.e. $t \in [a,b]_{\mathbb{T}}$ and for $x \in \overline{B}_r(A) := \{x \in \mathbb{R} : |x - A| \leq r\}$.

In this case, the function f is said to be the *generalized Δ-Carathéodory function.*

Definition 11.3.6. *A function $x : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is said to be a solution of generalized Δ-Carathéodory dynamic problem* (11.1) *if x satisfies the following conditions.*

(i) *x is Δ-ACG$_*$ function on each compact time scale subinterval $\mathbb{J}$ of $[a,b]_{\mathbb{T}}$.*

(ii) *$x^{\Delta} = f(t,x)$ Δ-a.e. $t \in [a,b]_{\mathbb{T}}$ and $x(a) = A$.*

The above condition (C3) can be written as

$$0 \le f(t,x) - g_r(t) \le h_r(t) - g_r(t) \quad (\Delta\text{-a.e. } t \in [a,b]_{\mathbb{T}} \text{ and } x \in \overline{B}_r(A)).$$

Then, $h_r - g_r$ is nonnegative Henstock–Kurzweil Δ-integrable function and by Theorem 2.19 of [7], it is a Δ-Lebesgue integrable function. Also, $f(t,x) - g_r(t)$ satisfies condition (C-iii). Thus, our generalized Δ-Carathéodory dynamic equation (11.1) is the essentially first-order dynamic equation of the form $x^{\Delta} = F(t,x) + G(t)$, where $F(t,x)$ is Δ-Lebesgue integrable function and $G(t)$ is Henstock–Kurzweil Δ-integrable function on $\mathbb{T}$, that is, *a first-order dynamic equation perturbed by Henstock–Kurzweil Δ-integrable function.*

11.3.2 Existence and uniqueness of solutions

Theorem 11.3.7. *If $f : [a,b]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ be a generalized Δ-Carathéodory function, then the dynamic problem* (11.1) *has generalized Δ-Carathéodory solution in $[a,b]_{\mathbb{T}}$.*

Proof. Since f is a generalized Δ-Carathéodory function, by condition (C3), there exist two Henstock–Kurzweil Δ-integrable functions g_r and h_r on $\mathbb{T}$ such that for all $x \in \overline{B}_r(A)$,

$$g_r(t) \le f(t,x) \le h_r(t) \quad \Delta\text{-a.e. } t \in \mathbb{T}.$$

The function $h_r - g_r$ is nonnegative and Henstock–Kurzweil Δ-integrable on $[a,b]_{\mathbb{T}}$, it follows that, $h_r - g_r$ is Δ-Lebesgue integrable on $[a,b]_{\mathbb{T}}$. Define $F : [a,b]_{\mathbb{T}} \times \overline{B}_r(A) \to \mathbb{R}$ by

$$F(t,x) := f\left(t, x + (\mathrm{HK})\int_a^t g_r(s)\Delta s\right) - g_r(t).$$

Then F is a Δ-Carathéodory function satisfying

$$0 \le F(t,x) \le h_r(t) - g_r(t) \text{ for all } (t,x) \in \Omega,$$

where $\Omega := \left\{(t,x) \in [a,b]_{\mathbb{T}} \times \overline{B}_r(A) : \left|x + (\mathrm{HK})\int_a^t g_r(\tau)\Delta\tau - A\right| \le r\right\}$. By

Theorem 11.2.3, there is a function $y : [a,b]_{\mathbb{T}} \to \mathbb{R}$ such that $y^{\Delta}(t) = F(t,y)$ Δ-a.e. $t \in [a,b]_{\mathbb{T}}$ and $y(a) = A$. Define

$$\phi(t) := y(t) + (\text{HK}) \int_a^t g_r(\tau)\Delta\tau \text{ for } t \in [a,b]_{\mathbb{T}}.$$

Then, for Δ-a.e. $t \in [a,b]_{\mathbb{T}}$,

$$\begin{aligned}\phi^{\Delta}(t) &= y^{\Delta}(t) + g_r(t)\\ &= F(t,y) + g_r(t)\\ &= f\left(t, y + (\text{HK}) \int_a^t g_r(\tau)\Delta\tau\right) - g_r(t) + g_r(t)\\ &= f\left(t, y + (\text{HK}) \int_a^t g_r(\tau)\Delta\tau\right)\end{aligned}$$

and $\phi(a) = y(a) = A$. Here, we note that the function ϕ is Δ-ACG$_*$ on $[a,b]_{\mathbb{T}}$ because y is Δ-ACG$_*$ on $[a,b]_{\mathbb{T}}$ and g_r is Henstock–Kurzweil Δ-integrable there. $\square$

Example 11.3.8. *Let* $\mathbb{T} := \left\{t = \frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$. *Let* $f(t,x) := h(t,x) + g(t)$, *where* $|h(t,x)| \leq H(t)$, $\forall\, t \in [0,1]_{\mathbb{T}}$ *and* $x \in \overline{B}_r(A)$ *and* $H(t)$ *is Lebesgue* Δ*-integrable on* $\mathbb{T}$. *Define* $g : \mathbb{T} \to \mathbb{R}$ *by*

$$g(t) := \begin{cases} (-1)^n\, n & \text{if } t = \frac{1}{n},\\ L & \text{if } t = 0,\end{cases}$$

where L is any constant. The function g is neither Δ-integrable nor Lebesgue Δ-integrable on $\mathbb{T}$. But, it is Henstock–Kurzweil Δ-integrable on $\mathbb{T}$. Note that $\forall\, t \in [0,1]_{\mathbb{T}}$ and $x \in \overline{B}_r(A)$, $g(t) - H(t) \leq f(t,x) \leq g(t) + H(t)$. Thus, $f(t,x)$ is generalized Δ-Carathéodory function and by Theorem 11.3.7 the generalized dynamic problem (11.1) *has a solution in $\mathbb{T}$. When $h(t,x) = 0$, the function $x : \mathbb{T} \to \mathbb{R}$ defined by*

$$x(t) := \begin{cases} 0 & \text{if } t = 1,\\ \displaystyle\sum_{k=2}^{n} \frac{(-1)^{k+1}}{k-1} & \text{if } t = \frac{1}{n},\\ -\ln(2) & \text{if } t = 0\end{cases}$$

is a solution of the generalized dynamic problem (11.1) *with f as defined above and $x(0) = A = -\ln(2)$.*

Theorem 11.3.9. *Let $f : [a,b]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ be a Δ-Carathéodory function. Define*

$$F_n(t) = (HK) \int_a^t \phi_n(\tau)\Delta\tau,$$

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where $\{\phi_n\}$ is a sequence of Δ-measurable simple functions. If $\{F_n\}$ is a sequence of Δ-UACG$_*$ functions on $[a,b]_{\mathbb{T}}$ such that $F_n \to x$ uniformly Δ-a.e. on $[a,b]_{\mathbb{T}}$, then x is generalized Δ-Carathéodory solution to the dynamic problem (11.1).

Proof. Since $f(t,x)$ is Δ-measurable in t, there exists a nondecreasing sequence of Δ-measurable simple functions $\{\phi_n\}$ on $[a,b]_{\mathbb{T}}$ such that $\phi_n(t) \to f(t,x(t))$ for all $t \in [a,b]_{\mathbb{T}}$. Since each ϕ_n is Henstock–Kurzweil Δ-integrable and $\phi_n(t) \leq \phi_{n+1}(t)$ Δ-a.e. on $[a,b]_{\mathbb{T}}$ and for all $n \in \mathbb{N}$, by monotone convergence theorem, it follows that $f(t,x(t))$ is Henstock–Kurzweil Δ-integrable on $[a,b]_{\mathbb{T}}$ and

$$(\mathrm{HK})\int_a^t f(\tau,x(\tau))\Delta\tau = \lim_{n\to\infty}(\mathrm{HK})\int_a^t \phi_n(\tau)\Delta\tau.$$

Define

$$F_n(t) := (\mathrm{HK})\int_a^t \phi_n(\tau)\Delta\tau + A, \quad \text{for } t \in [a,b]_{\mathbb{T}}.$$

Then, $\{F_n\}$ is a sequence of Δ-UACG$_*$ functions and $F_n(t) \to x(t)$, $t \in [a,b]_{\mathbb{T}}$. Therefore

$$\lim_{n\to\infty} F_n(t) = (\mathrm{HK})\int_a^t f(\tau,x(\tau))\Delta\tau + A.$$

This gives

$$x(t) = (\mathrm{HK})\int_a^t f(\tau,x(\tau))\Delta\tau + A, \quad t \in [a,b]_{\mathbb{T}}$$

which gives the dynamic problem (11.1). Now, since $\{F_n\}$ is a sequence of Δ-UACG$_*$ functions converges to x and x is a primitive of Henstock–Kurzweil Δ-integrable function f on $[a,b]_{\mathbb{T}}$, it follows that x is Δ-ACG$_*$ on $[a,b]_{\mathbb{T}}$. □

Below we give an inequality which is useful for further analysis.

Theorem 11.3.10. *Suppose $g : [0,\infty) \to [0,\infty)$ is continuous nondecreasing function and $y : [a,b]_{\mathbb{T}} \to [0,\infty)$ is bounded function such that $g \circ y : [a,b]_{\mathbb{T}} \to [0,\infty)$ is rd-continuous. If $G : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a strictly increasing function such that*

$$G^{\Delta}(u) = \frac{1}{g(u)} \neq 0 \text{ for } u \in \mathbb{T},$$

then for all $t \in [a,b]_{\mathbb{T}}$ and for some constant $l \geq 0$

$$y(t) \leq l + (HK)\int_a^t g(y(\tau))\Delta\tau \tag{11.8}$$

implies

$$y(t) \leq G^{-1}(G^{\sigma}(l) + t - a),$$

where $G^{-1} : \widetilde{\mathbb{T}} = G([a,b]_{\mathbb{T}}) \to [a,b]_{\mathbb{T}}$ is the inverse of $G : [a,b]_{\mathbb{T}} \to \mathbb{R}$.

Proof. Assume for all $t \in [a,b]_{\mathbb{T}}$,

$$y(t) \leq l + (\mathrm{HK}) \int_a^t g(y(\tau))\Delta\tau$$

for some constant $l \geq 0$. Denote $G(a) = \alpha$, $G(b) = \beta$, where $\alpha = \min \widetilde{\mathbb{T}}$.

Let $T \in [a,b]_{\mathbb{T}}$ be such that $\alpha - (t-a) < G(T) < \beta - (t-a)$ for all $t \in [a,b]_{\mathbb{T}}$. For $s \in [\alpha - G(T), \beta - G(T)]_{\widetilde{\mathbb{T}}}$, set $G^{-1}(G(T)+s) = u$. Then, $u \in [a,b]_{\mathbb{T}}$ and by hypothesis

$$G^{\Delta}(G^{-1}(G(T)+s)) = \frac{1}{g(G^{-1}(G(T)+s))}.$$

But, by derivative of the inverse, we have $(G^{-1}(p))^{\Delta} = (\frac{1}{G^{\Delta}} \circ G^{-1})(p)$ for $p \in [\alpha,\beta]_{\widetilde{\mathbb{T}}}$. So, $(G^{-1}(p))^{\Delta} = (g \circ G^{-1})(p)$. Now, for $t \in [a,b]_{\mathbb{T}}$

$$\begin{aligned}
&(\mathrm{HK}) \int_a^t g(G^{-1}(G(T)+\tau-a))\Delta\tau \\
&= (\mathrm{HK}) \int_a^t (g \circ G^{-1})(G(T)+\tau-a)\Delta\tau \\
&= (\mathrm{HK}) \int_a^t (G^{-1})^{\Delta}(G(T)+\tau-a)\Delta\tau \\
&= G^{-1}(G(T)+t-a) - G^{-1}(G(T)+a-a) \\
&= G^{-1}(G(T)+t-a) - T.
\end{aligned}$$

Hence,

$$T + (\mathrm{HK}) \int_a^t g(G^{-1}(G(T)+\tau-a))\Delta\tau = G^{-1}(G(T)+t-a).$$

Now, let $T = \sigma(l)$ for some $l \in [a,b]_{\mathbb{T}}^{\kappa}$ such that $G^{\sigma}(l) < \alpha - b + a$. Then,

$$\sigma(l) + (\mathrm{HK}) \int_a^t g(G^{-1}(G^{\sigma}(l)+\tau-a))\Delta\tau = G^{-1}(G^{\sigma}(l)+t-a). \tag{11.9}$$

Since G^{-1} is bounded, combing (11.8) and (11.9), we obtain

$$\begin{aligned}
&y(t) - G^{-1}(G^{\sigma}(l)+t-a) \\
&\leq l + (\mathrm{HK}) \int_a^t g(y(\tau))\Delta\tau - \sigma(l) - (\mathrm{HK}) \int_a^t g(G^{-1}(G^{\sigma}(l)+\tau-a))\Delta\tau \\
&= l - \sigma(l) + (\mathrm{HK}) \int_a^t [g(y(\tau)) - g(G^{-1}(G^{\sigma}(l)+\tau-a))]\Delta\tau.
\end{aligned}$$

Since g is nondecreasing, and y and G^{-1} are bounded, we arrive at

$$|g(y(\tau)) - g(G^{-1}(G^{\sigma}(l)+\tau-a))| < M$$

for some $M > 0$ and for all $\tau \in [a,b]_{\mathbb{T}}$. Therefore, for $t \in [a,b]_{\mathbb{T}}$

$$y(t) - G^{-1}\big(G^{\sigma}(l) + t - a\big) \leq l - \sigma(l) + M(t-a).$$

If $\Lambda := \{t \in [a,b]_{\mathbb{T}} : y(t) < G^{-1}(G^{\sigma}(l) + \tau - a) \text{ for } a \leq \tau \leq b\}$, then one can see that $\sup \Lambda = \max \mathbb{T} = b$. Hence, $y(t) \leq G^{-1}\big(G^{\sigma}(l) + t - a\big)$ for $t \in [a,b]_{\mathbb{T}}$. □

The local uniqueness of solution of dynamic problem (11.1) involving an Osgood-type condition is given below.

Theorem 11.3.11. *Assume that* $f : [a,b]_{\mathbb{T}} \times \overline{B}_r(A) \to \mathbb{R}$ *satisfy the condition*

$$|f(t,x) - f(t,y)| \leq g(|x-y|),$$

where $g : [a,b]_{\mathbb{T}} \to [0,\infty)$ *is such that its extension* $\overline{g} : [0,\infty) \to [0,\infty)$ *is an increasing function with* $\overline{g}(0) = 0$ *and* $\overline{g}(s) > 0$ *for* $s \in (0,\infty)$ *and for every* $k > 0$,

$$(HK)\int_0^k \frac{1}{\overline{g}(s)}ds = \infty.$$

Then, the generalized Δ*-Carathéodory dynamic problem* (11.1) *has unique solution in* $[a,b]_{\mathbb{T}}$.

Proof. Assume that x and y are two solutions of the generalized Δ-Carathéodory dynamic problem (11.1). Then, $x, y : [a,b]_{\mathbb{T}} \to \mathbb{R}$ are such that

$$x(t) = A + (\mathrm{HK})\int_a^t f(\tau, x(\tau))\Delta\tau$$

and

$$y(t) = A + (\mathrm{HK})\int_a^t f(\tau, y(\tau))\Delta\tau.$$

If $z(t) := x(t) - y(t)$ for all $t \in [a,b]_{\mathbb{T}}$, then

$$\begin{aligned}
|z(t)| &= \left|(\mathrm{HK})\int_a^t [f(\tau,x(\tau)) - f(\tau,y(\tau))]\Delta\tau\right| \\
&\leq (\mathrm{HK})\int_a^t |f(\tau,x(\tau)) - f(\tau,y(\tau))|\,\Delta\tau \\
&\leq (\mathrm{HK})\int_a^t g(|z(\tau)|)\Delta\tau \\
&\leq (\mathrm{HK})\int_a^b g(|z(\tau)|)\Delta\tau.
\end{aligned}$$

If $G^{\Delta}(u) = \dfrac{1}{g(u)}$, then by Theorem 11.3.10, we can write

$$\begin{aligned} |z(t)| &\leq G^{-1}(G^{\sigma}(0) + b - a) \\ G(|z(t)|) &\leq G^{\sigma}(0) + b - a \\ G(|z(t)|) - G^{\sigma}(0) &\leq b - a \\ (\mathrm{HK}) \int_{\sigma(0)}^{|z(t)|} G^{\Delta}(r)\Delta r &\leq b - a \\ (\mathrm{HK}) \int_{\sigma(0)}^{|z(t)|} \frac{1}{g(r)}\Delta r &\leq b - a. \end{aligned}$$

But, from Theorems 19 and 20 of [13],

$$(\mathrm{HK}) \int_{\sigma(0)}^{|z(t)|} \frac{1}{g(r)}\Delta r = (\mathrm{HK}) \int_{0}^{|z(t)|} \frac{1}{\overline{g}(s)} ds.$$

Therefore,

$$(\mathrm{HK}) \int_{0}^{|z(t)|} \frac{1}{\overline{g}(s)} ds \leq b - a$$

which gives $\infty \leq b - a$, a contradiction. Hence, we must have $|z(t)| = 0$ and therefore the dynamic problem (11.1) has unique solution in $[a, b]_{\mathbb{T}}$. □

Remark 11.3.12. *If the conditions given in Theorem 11.3.11 do not hold, then the dynamic problem* (11.1) *may have more than one solutions. This is shown in the following example:*

Let $\mathbb{T} := \left\{ t = \dfrac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}$ *and define* $f : [0,1]_{\mathbb{T}} \times \overline{B}_r(0) \to \mathbb{R}$*by*

$$f(t,x) := \begin{cases} \dfrac{1}{2(\sqrt{t} + \sqrt{\sigma(t)})} & \text{for } x < t, \\ \dfrac{1}{(\sqrt{t} + \sqrt{\sigma(t)})} & \text{for } x \geq t, \end{cases}$$

where $\overline{B}_r(0) = \{x \in \mathbb{R} : |x| \leq r\}$. *Then, we find that*

$$(\mathrm{HK}) \int_0^1 f(t, x(t))\Delta t = 1.$$

So, f does not satisfy conditions given in Theorem 11.3.11. In fact, $x_1(t) = \sqrt{t}$ *and* $x_2(t) = \dfrac{\sqrt{t}}{2}$ *are two solutions of dynamic problem* (11.1).

Remark 11.3.13. *The conditions given in the Theorem 11.3.11 do not guarantee the existence of a solution of the dynamic problem* (11.1), *as shown in the next example:*

Let $[0,1)_{\mathbb{T}}$ be a time scale interval that contains a countable infinite subset $\cup_{i=1}^{\infty}\{r_i\}$ with $\sigma(r_i) = r_i$. Let $f : [0,1]_{\mathbb{T}} \times \overline{B}_r(0) \to \mathbb{R}$ defined by

$$f(t,x) := \begin{cases} 0 & \text{for } x \neq 0,\ t \neq r_i \\ 1 & \text{for } x \neq 0,\ t = r_i \quad \text{or } x = 0,\ t \neq r_i, \\ 2 & \text{for } x = 0,\ t = r_i. \end{cases}$$

This f is generalized Δ-Carathéodory function and satisfy condition given in Theorem 11.3.11 with $\overline{g} \equiv 0$. For this function f, dynamic problem (11.1) has no solution.

11.4 Dependency and Convergence of Solutions

Theorem 11.4.1. *Assume that $f : [a,b]_{\mathbb{T}} \times \overline{B}_r(A) \to \mathbb{R}$ satisfy the condition*

$$|f(t,x) - f(t,y)| \leq g(|x-y|),$$

where $g : [a,b]_{\mathbb{T}} \to [0,\infty)$ is an increasing continuous function such that $\frac{1}{g(u)} \neq 0$ for $u \in [a,b]_{\mathbb{T}}$. Let x and y be such that $x^{\Delta}(t) = f(t,x)$, $x(a) = A$ and $y^{\Delta}(t) = f(t,y)$, $y(a) = B$. Then

$$|x(t) - y(t)| \leq G^{-1}(G(|A-B|) + t - a) \quad \text{for all } t \in [a,b]_{\mathbb{T}},$$

where $G : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is a strictly increasing function such that $G^{\Delta}(u) = \dfrac{1}{g(u)}$ for all $u \in [a,b]_{\mathbb{T}}$.

Proof. By hypothesis, we have

$$x(t) = A + (\text{HK})\int_a^t f(\tau, x(\tau))\Delta\tau \quad \text{and} \quad y(t) = B + (\text{HK})\int_a^t f(\tau, y(\tau))\Delta\tau.$$

Then for all $t \in [a,b]_{\mathbb{T}}$,

$$\begin{aligned} |x(t) - y(t)| &\leq |A-B| + (\text{HK})\int_a^t |f(\tau, x(\tau)) - f(\tau, y(\tau))|\,\Delta\tau \\ &\leq |A-B| + (\text{HK})\int_a^t g\left(|x(\tau) - y(\tau)|\right)\Delta\tau. \end{aligned}$$

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By Theorem 11.3.10, we have

$$|x(t) - y(t)| \leq G^{-1}\left(G(|A - B|) + t - a\right) \quad \text{for all } t \in [a, b]_{\mathbb{T}},$$

which shows that the solutions of dynamic problem (11.1) depend continuously on the initial condition. To emphasize the dependence on the initial condition, we denote the solutions of (11.1) by $\phi(t, a, A)$. Thus,

$$|\phi(t, a, A) - \phi(t, a, B)| \leq G^{-1}\left(G(|A - B|) + t - a\right)$$

for all $t \in [a, b]_{\mathbb{T}}$. □

Corollary 11.4.2. *Under the assumption of the above theorem we have the following.*

$$\begin{aligned} |\phi(t, a, A) - \phi(s, \bar{a}, B)| \leq\ & G^{-1}\left(G(|A - B|) + t - a\right) \\ & + G^{-1}\left(G(M\,|\bar{a} - a|) + t - \bar{a}\right) + M\,|t - s|, \end{aligned}$$

where $M := \sup_{t \in \mathbb{T}} |f(t, \bar{a}, B)|$ *and* $\bar{a} \in (a, b]_{\mathbb{T}}$.

Proof. We write

$$\begin{aligned} |\phi(t, a, A) - \phi(s, \bar{a}, B)| \leq\ & |\phi(t, a, A) - \phi(t, a, B)| + |\phi(t, a, B) - \phi(t, \bar{a}, B)| \\ & + |\phi(t, \bar{a}, B) - \phi(s, \bar{a}, B)|. \end{aligned} \tag{11.10}$$

Consider

$$\begin{aligned} & |\phi(t, a, A) - \phi(t, a, B)| \\ & \leq |A - B| + (\mathrm{HK})\int_a^t |f(\tau, \phi(\tau, a, A)) - f(\tau, \phi(\tau, a, B))|\,\Delta\tau \\ & \leq |A - B| + (\mathrm{HK})\int_a^t g\left(|\phi(\tau, a, A) - \phi(\tau, a, B)|\right)\Delta\tau, \end{aligned}$$

which yields

$$|\phi(t, a, A) - \phi(t, a, B)| \leq G^{-1}\left(G(|A - B|) + t - a\right). \tag{11.11}$$

Assuming $0 \leq a < \bar{a} < t$,

$$\begin{aligned}
&|\phi(t,a,B) - \phi(t,\bar{a},B)| \\
&\leq (\mathrm{HK}) \int_a^t |f(\tau,\phi(\tau,a,B)) - f(\tau,\phi(\tau,\bar{a},B))|\, \Delta\tau \\
&= (\mathrm{HK}) \int_a^{\bar{a}} |f(\tau,\phi(\tau,a,B)) - f(\tau,\phi(\tau,\bar{a},B))|\, \Delta\tau \\
&\quad + (\mathrm{HK}) \int_{\bar{a}}^t |f(\tau,\phi(\tau,a,B)) - f(\tau,\phi(\tau,\bar{a},B))|\, \Delta\tau \\
&\leq (\mathrm{HK}) \int_a^{\bar{a}} |f(\tau,\phi(\tau,a,B))|\, \Delta\tau \\
&\quad + (\mathrm{HK}) \int_{\bar{a}}^t |f(\tau,\phi(\tau,a,B)) - f(\tau,\phi(\tau,\bar{a},B))|\, \Delta\tau \\
&\leq M\,|\bar{a}-a| + (\mathrm{HK}) \int_{\bar{a}}^t |f(\tau,\phi(\tau,a,B)) - f(\tau,\phi(\tau,\bar{a},B))|\, \Delta\tau \\
&\leq M\,|\bar{a}-a| + (\mathrm{HK}) \int_{\bar{a}}^t g\left(|\phi(\tau,a,B) - \phi(\tau,\bar{a},B)|\right) \Delta\tau,
\end{aligned}$$

which yields

$$|\phi(t,a,B) - \phi(t,\bar{a},B)| \leq G^{-1}\left(G(M\,|\bar{a}-a|) + t - \bar{a}\right). \tag{11.12}$$

Now,

$$\begin{aligned}
|\phi(t,\bar{a},B) - \phi(s,\bar{a},B)| &= \left|(\mathrm{HK}) \int_s^t f(\tau,\bar{a},B)\Delta\tau\right| \\
&\leq (\mathrm{HK}) \int_s^t |f(\tau,\bar{a},B)|\Delta\tau.
\end{aligned}$$

That is,

$$|\phi(t,\bar{a},B) - \phi(s,\bar{a},B)| \leq M\,|t-s|, \tag{11.13}$$

Hence, using (11.11), (11.12) and (11.13), (11.10) becomes

$$\begin{aligned}
|\phi(t,a,A) - \phi(s,\bar{a},B)| &\leq G^{-1}\left(G(|A-B|) + t - a\right) \\
&\quad + G^{-1}\left(G(M\,|\bar{a}-a|) + t - a\right) + M\,|t-s|.
\end{aligned} \tag{11.14}$$

□

Now, to investigate the convergence of solutions of dynamic problem (11.1), we provide an analogous concept of equi-integrability and a related basic result.

Definition 11.4.3. *A family of functions $\{f_n\}$ is said to be Henstock-Kurzweil Δ-equi-integrable on $[a,b]_{\mathbb{T}}$, if for every $\varepsilon > 0$, there exists a Δ-gauge,*

δ for $[a,b]_{\mathbb{T}}$ such that for all $n \in \mathbb{N}$

$$\left|(HK)\int_a^b f_n - \sum_{i=1}^k f_n(\xi_i)(t_i - t_{i-1})\right| < \varepsilon$$

for all δ-fine partitions $\mathcal{P} = \{\xi_i, [t_{i-1}, t_i]_{\mathbb{T}}\}_{i=1}^k$ of $[a,b]_{\mathbb{T}}$.

Theorem 11.4.4. *If $\{f_n\}$ is a sequence of Henstock–Kurzweil Δ-equi-integrable functions on $[a,b]_{\mathbb{T}}$ such that $f_n \to f$ on $[a,b]_{\mathbb{T}}$, then f is Henstock–Kurzweil Δ-integrable on $[a,b]_{\mathbb{T}}$ and*

$$\lim_{n\to\infty}(HK)\int_a^b f_n(t)\Delta t = (HK)\int_a^b f(t)\Delta t.$$

Proof. Since $\{f_n\}$ is sequence of Henstock–Kurzweil Δ-equi-integrable functions on $[a,b]_{\mathbb{T}}$, for every $\varepsilon > 0$, there is a Δ-gauge, δ for $[a,b]_{\mathbb{T}}$ such that for all $n \in \mathbb{N}$

$$\left|(\mathrm{HK})\int_a^b f_n - \sum_{i=1}^k f_n(\xi_i)(t_i - t_{i-1})\right| < \varepsilon$$

for all δ-fine partitions $\mathcal{P} = \{\xi_i, [t_{i-1}, t_i]_{\mathbb{T}}\}_{i=1}^k$ of $[a,b]_{\mathbb{T}}$.

We claim that the sequence $\left\{(HK)\int_a^b f_n(t)\Delta t\right\}$ is convergent. Consider

$$\begin{aligned}
&\left|(HK)\int_a^b f_n(t)\Delta t - (HK)\int_a^b f_m(t)\Delta t\right| \\
&= \left|(HK)\int_a^b f_n(t)\Delta t - \sum_{i=1}^k f_n(\xi_i)(t_i - t_{i-1}) + \sum_{i=1}^k f_n(\xi_i)(t_i - t_{i-1})\right. \\
&\qquad \left. - \sum_{i=1}^k f_m(\xi_i)(t_i - t_{i-1}) + \sum_{i=1}^k f_m(\xi_i)(t_i - t_{i-1}) - (HK)\int_a^b f_m(t)\Delta t\right| \\
&\le \left|(HK)\int_a^b f_n(t)\Delta t - \sum_{i=1}^k f_n(\xi_i)(t_i - t_{i-1})\right| \\
&\qquad + \left|\sum_{i=1}^k f_n(\xi_i)(t_i - t_{i-1}) - \sum_{i=1}^k f_m(\xi_i)(t_i - t_{i-1})\right| \\
&\qquad + \left|\sum_{i=1}^k f_m(\xi_i)(t_i - t_{i-1}) - (HK)\int_a^b f_m(t)\Delta t\right| \\
&< \varepsilon + \varepsilon + \left|\sum_{i=1}^k f_n(\xi_i)(t_i - t_{i-1}) - \sum_{i=1}^k f_m(\xi_i)(t_i - t_{i-1})\right| \\
&= 2\varepsilon + \sum_{i=1}^k \left|f_n(\xi_i) - f_m(\xi_i)\right| \left|t_i - t_{i-1}\right|.
\end{aligned}$$

Since $f_n \to f$ on $[a,b]_{\mathbb{T}}$, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m,\ n \geq n_0$ and for all $\xi_i \in [a,b]_{\mathbb{T}}$, we have

$$\left|f_n(\xi_i) - f_m(\xi_i)\right| < \frac{\varepsilon}{(b-a)}.$$

Then

$$\begin{aligned}\left|(HK)\int_a^b f_n(t)\Delta t - (HK)\int_a^b f_m(t)\Delta t\right| &< 2\varepsilon + \sum_{i=1}^{k} \frac{\varepsilon}{(b-a)}|t_i - t_{i-1}| \\ &< 2\varepsilon + \varepsilon = 3\varepsilon.\end{aligned}$$

This implies that $\left\{(HK)\int_a^b f_n(t)\Delta t\right\}$ is a Cauchy sequence in $\mathbb{R}$ and hence it is convergent. Let

$$\lim_{n\to\infty}(HK)\int_a^b f_n(t)\Delta t = L.$$

Then

$$\begin{aligned}&\left|\sum_{i=1}^{k} f(\xi_i)(t_i - t_{i-1}) - L\right| \\ &= \left|\sum_{i=1}^{k} f(\xi_i)(t_i - t_{i-1}) - \sum_{i=1}^{k} f_n(\xi_i)(t_i - t_{i-1}) + \sum_{i=1}^{k} f_n(\xi_i)(t_i - t_{i-1}) - L\right| \\ &\leq \left|\sum_{i=1}^{k} f(\xi_i)(t_i - t_{i-1}) - \sum_{i=1}^{k} f_n(\xi_i)(t_i - t_{i-1})\right| + \left|\sum_{i=1}^{k} f_n(\xi_i)(t_i - t_{i-1}) - L\right| \\ &< \varepsilon + \varepsilon = 2\varepsilon.\end{aligned}$$

This yields that

$$(HK)\int_a^b f(t)\Delta t = L.$$

Hence, f is Henstock–Kurzweil Δ-integrable on $\mathbb{T}$ and

$$\lim_{n\to\infty}(HK)\int_a^b f_n(t)\Delta t = (HK)\int_a^b f(t)\Delta t.$$

□

Using the above result we have the following.

Theorem 11.4.5. *Assume that $\{f_n\}$ is a sequence of Henstock–Kurzweil Δ-equi-integrable functions on $[a,b]_{\mathbb{T}}$ such that $f_n \to f$ on $[a,b]_{\mathbb{T}}$ and for each $n \in \mathbb{N}$, $f_n : [a,b]_{\mathbb{T}} \times \overline{B}_r(A) \to \mathbb{R}$ satisfy*

$$|f_n(t,x) - f_n(t,y)| \leq g(|x-y|),$$

where $g : [a,b]_{\mathbb{T}} \to [0,\infty)$ is a continuous increasing function with $g(0) = 0$ and $g(t) > 0$ for $t \in [a,b]_{\mathbb{T}}$. If, for each $n \in \mathbb{N}$, x_n is a solution of

$$\begin{cases} x_n^{\Delta}(t) = f_n(t, x_n), & t \in [a,b]_{\mathbb{T}}^{\kappa}; \\ x_n(a) = A, \end{cases} \tag{11.15}$$

then there is a subsequence $\{y_n\}$ of $\{x_n\}$ and a function $x : [a,b]_{\mathbb{T}} \to \mathbb{R}$ such that $y_n \to x$ and x is solution of dynamic problem (11.1) in $[a,b]_{\mathbb{T}}$.

Proof. From the hypothesis, we have

$$x_n(t) = x_n(a) + (HK)\int_a^t f_n(\tau, x_n)\Delta\tau,$$

which yields

$$|x_n(t) - x_n(a)| \le (HK)\int_a^t |f_n(\tau, x_n)|\,\Delta\tau. \tag{11.16}$$

Now,

$$\begin{aligned} |f_n(\tau, x_n(\tau))| &= |f_n(\tau, x_n(a)) + f_n(\tau, x_n(\tau)) - f_n(\tau, x_n(a))| \\ &\le |f_n(\tau, x_n(a))| + |f_n(\tau, x_n(\tau)) - f_n(\tau, x_n(a))| \\ &\le |f_n(\tau, x_n(a))| + g\left(|x_n(\tau) - x_n(a)|\right). \end{aligned}$$

Then

$$(HK)\int_a^t |f_n(\tau, x_n(\tau))|\,\Delta\tau \le M + (HK)\int_a^t g\left(|x_n(\tau) - x_n(a)|\right)\Delta\tau,$$

where $M := (HK)\displaystyle\int_a^b |f_n(\tau, x_n(a))|\,\Delta\tau$. By Theorem 11.3.10, we have

$$(HK)\int_a^t |f_n(\tau, x_n(\tau))|\,\Delta\tau \le G^{-1}(G(M) + t - a),$$

where $G^{\Delta}(u) = \dfrac{1}{g(u)}$. So, equation (11.16) becomes

$$|x_n(t) - x_n(a)| \le G^{-1}(G(M) + t - a).$$

Therefore

$$\begin{aligned} |x_n(t)| &\le |x_n(t) - x_n(a)| + |x_n(a)| \\ &\le G^{-1}\left(G(M) + t - a\right) + x_0 \\ &= \alpha, \end{aligned}$$

where $\alpha := G^{-1}(G(M) + b - a) + x_0$. Therefore, $\{x_n\}$ is uniformly bounded on $[a,b]_{\mathbb{T}}$.

Now, for given any $\varepsilon > 0$, we can find $\delta > 0$ such that if $|t - t_1| < \delta$, then

$$\begin{aligned}|x_n(t) - x_n(t_1)| &= \left| x_n(a) + (HK)\int_a^t f_n(\tau, x_n(\tau))\Delta\tau - x_n(a) \right.\\ &\quad \left. - (HK)\int_a^{t_1} f_n(\tau, x_n(\tau))\Delta\tau \right| \\ &= \left| (HK)\int_{t_1}^t f_n(\tau, x_n(\tau))\Delta\tau \right| \\ &\le (HK)\int_{t_1}^t |f_n(\tau, x_n(\tau))|\,\Delta\tau \\ &\le L\,|t - t_1| < L\delta.\end{aligned}$$

That is,

$$|x_n(t) - x_n(t_1)| < \varepsilon,$$

where $\varepsilon := L\delta$. Therefore $\{x_n\}$ is equicontinuous on $[a,b]_{\mathbb{T}}$. By Arzelá–Ascoli's theorem [1], there exists a subsequence $\{y_n\}$ of $\{x_n\}$ such that $y_n \to x$ uniformly on $\mathbb{T}$. Also, by hypothesis, we have

$$y_n(t) = A + (HK)\int_a^t f_n(\tau, y_n(\tau))\Delta\tau.$$

Since $f_n \to f$ on $[a,b]_{\mathbb{T}}$ and $\{f_n\}$ is Henstock–Kurzweil Δ-equi-integrable on $\mathbb{T}$, by Theorem 11.3.10, it follows that

$$\begin{aligned}\lim_{n\to\infty} y_n(t) &= A + \lim_{n\to\infty}(HK)\int_a^t f_n\big(\tau, y_n(\tau)\big)\Delta\tau \\ x(t) &= A + (HK)\int_a^t \lim_{n\to\infty} f_n\big(\tau, y_n(\tau)\big)\Delta\tau \\ x(t) &= A + (HK)\int_a^t f\big(\tau, x(\tau)\big)\Delta\tau.\end{aligned}$$

That is,

$$\begin{cases} x^{\Delta}(t) = f\big(t, x(t)\big), \ t \in \mathbb{T}, \\ x(a) = A. \end{cases}$$

□

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