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Ulam Type Stability Results of Nonlinear Impulsive Volterra–Fredholm Integro-Dynamic Adjoint Equations on Time Scale

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Abstract: This paper is dedicated to exploring the existence, uniqueness and Ulam stability analysis applied to a specific class of mathematical equations known as nonlinear impulsive Volterra Fredholm integro-dynamic adjoint equations within finite time scale intervals. The primary aim is to establish sufficient conditions that demonstrate Ulam stability for this particular class of equations on the considered time scales. The research methodology relies on the Banach contraction principle, Picard operator and extended integral inequality applicable to piecewise continuous functions on time scales. To illustrate the applicability of the findings, an example is provided.

Keywords: Volterra integral; existence; uniqueness; time scale; Hyers–Ulam stability; Hyers–Ulam–Rassias stability; impulses

MSC: 34N05; 34D20; 34A37; 45J05

1. Introduction

The utilization of impulses in conjunction with differential equations has found significant composition in mathematical modeling. In our daily lives, we encounter a diverse array of phenomena and processes that experience immediate system alterations at certain instances. These instances are characterized by impulsive effects, which represent temporary disruptions within the system. Many authors have diligently examined the combination of differential equations and impulses, making substantial contributions to this field of study. Interested readers can explore the valuable insights provided by these papers [1–3].

One of the most representative fields in mathematical sciences is stability analysis [4,5] and it has many kinds but the interesting and important one is the Hyers–Ulam (HU) stability. The HU-type stability problem was first raised by Ulam [6,7] and solved partially for the Banach space case by Hyers [8]. Rassias generalized this concept [9] in 1978 and it was named HU–Rassias (HUR) stability. The notion of Ulam stability is both exciting and warrants careful consideration, as it serves to bridge the gap between the exact solution and an approximated solution when studying a given equation [10]. Exploring the concept of Ulam stability involves delving into an accomplished mathematical inquiry with diverse methodologies and approaches and one of these approaches involves the application of inequalities. This approach stands out for its advantages, notably its ability to work with fewer constraints and its relative simplicity compared to alternative methods. The authors, led by Wang, have achieved stability in terms of HU and HUR results for impulsive differential equations, as indicated in their references [11–13]. Li and Shen [14] have explored a distinct approach to examine the HU stability of linear differential equations of

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the second order. Numerous papers have been published representing the HU and HUR stability concepts and the reader is referred to references [15–18].

Time scale (TS) analysis has experienced a swift and notable rise in prominence, capturing considerable attention and generating substantial interest among researchers. Hilger is credited with the introduction of the theory of TS in his PhD thesis in 1988 [19]. The primary significance of TS theory lies in its ability to be employed in both differential equations and difference equations. It introduces a united mathematical framework that bridges the gap between difference equations and differential equations. This framework proves to be versatile and valuable in modeling situations where time phenomena exhibit a hybrid nature, combining both continuous and discrete elements seamlessly. This versatility allows researchers to utilize TS theory in various mathematical scenarios, expanding its applicability beyond traditional differential equations. For further details on the TS concept, see references [20–22].

There are many research publications addressing the qualitative properties of integro-dynamic systems with impulses on TSs. Zada et al. [23] investigated the HU stability and the HUR stability of Volterra integro-dynamic system of nonlinear form with delay and instantaneous impulses on TSs. Shah and Zada [24] studied the stability results of integro-dynamic systems (mixed) with instantaneous as well as noninstantaneous impulses on TSs. Then, Shah et al. [25] generalized these results to the Ulam-type Bielecki stabilities of the Hammerstein and integro-dynamic systems (mixed) of nonlinear form on TSs with instantaneous impulses. Bohner et al. [26,27] discussed some results regarding the first-order nonlinear dynamic initial value problems and nonlinear integro-dynamic equations. Recently, Scindia et al. [28] have published noteworthy findings concerning the Ulam stability of first-order nonlinear impulsive dynamic equations. They have achieved their results by proving the extended integral inequality on TSs.

To the best of our knowledge, there is no work available in the literature that addresses the Ulam-type stability properties of nonlinear impulsive Volterra Fredholm integro-dynamic (NIVFID) adjoint equations on TS. So, motivated by the results of reference [28] and the above-mentioned work, the main contribution in this paper is to investigate the existence, uniqueness, HU stability, generalized HU stability, HUR stability, and generalized HUR stability of the solution of a NIVFID adjoint equation, of the following form:

$$\begin{cases} \Phi^{\Delta}(\mathrm{r}) + \mathrm{p}(\mathrm{r})\Phi^{\sigma}(\mathrm{r}) = \int_{\mathrm{r}_0}^{\mathrm{r}} f(s,\Phi(s))\Delta s + \int_a^b f(s,\Phi(s))\Delta s, \quad \mathrm{r} \in \mathbb{J}^z \backslash \{\mathrm{r}_i\}, \quad i = \overline{1,\ m}, \\ \Phi(\mathrm{r}_k^+) - \Phi(\mathrm{r}_k^-) = \Upsilon_k(\Phi(\mathrm{r}_k^-)), \quad k = \overline{1,\ m}, \\ \Phi(\mathrm{r}_0) = \Phi_0 \in \mathbb{R}, \end{cases} \tag{1}$$

where $\mathbb{J} := [\mathrm{r}_0, \mathrm{r}_f]_{\mathbb{T}}$, $\mathbb{T}$ is a TS, $\mathbb{J}^z$ represents the derived form of $\mathbb{J}$, $0 \leq \mathrm{r}_0 = s_0 < s < \mathrm{r}_f$, $a, b \in \mathbb{R}$, $\overline{1,\ m}$ denotes $1, 2, 3, \ldots, m$, $\Phi \colon \mathbb{J} \to \mathbb{R}$ is a function that is currently unknown and needs to be determined, Φ^{Δ} is the delta derivative of Φ on TS and we define Φ^{σ} as the composition of functions, where $\Phi^{\sigma} = \Phi \circ \sigma$. Additionally, we have a function p mapping from the set $\mathbb{T}$ to $\mathbb{R}$, which is both positively regressive and rd-continuous. Furthermore, the function f, which takes inputs from the Cartesian product of $\mathbb{J}$ and $\mathbb{R}$ and returns values in $\mathbb{R}$, is right dense (rd)-continuous with respect to its first variable, and continuous with respect to its second variable; $\int_{\mathrm{r}_0}^{\mathrm{r}} f(s,\Phi(s))\Delta s$ represents the Volterra integral of f and $\int_a^b f(s,\Phi(s))\Delta s$ represents the Fredholm integral of f and $\Phi(\mathrm{r}_k^+) = \lim_{\tau \to 0^+} \Phi(\mathrm{r}_k + \tau)$ and $\Phi(\mathrm{r}_k^-) = \lim_{\tau \to 0^-} \Phi(\mathrm{r}_k - \tau)$ exist at r_k, satisfying

$$\mathrm{r}_0 < \mathrm{r}_1 < \mathrm{r}_2 < \mathrm{r}_3 < \mathrm{r}_4 < \cdots < \mathrm{r}_m < \mathrm{r}_{m+1} = \mathrm{r}_f < +\infty,$$

where each r_k represents a priori known moments of impulse. The function $\Upsilon_k \colon \mathbb{R} \to \mathbb{R}$ describes the discontinuity of Φ at each r_k. The relationship between r and r_k is that r is the independent variable and the above equation depends on it over the whole interval $\mathbb{J}$, while r_k is involved only in the impulses. In the entirety of our article, we make the assumption that the set TS does not comprise a subset of integers. Furthermore, we specify

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that, when dealing with isolated points, any impulses associated with them are regarded as having a zero value.

The outline of the paper is as follows. Section 2 outlines the preliminary results, as well as some basic concepts and remarks. Section 3 contains the existence and uniqueness results of (1). Section 4 comprises the stability results of (1). The achieved results are verified using an illustrative example in Section 5.

The following notations are used throughout the paper:

Symbol	Interpretation
$\mathbb{T}$	The set of time scale
$\mathbb{R}$	The set of real numbers
σ	Forward jump operator
ρ	Backward jump operator
μ	Graininess function
$\mathcal{C}_{rd}(\mathbb{T}, \mathbb{R})$	The set of right-dense-continuous functions
$\mathcal{R}(\mathbb{T})$	The set of regressive functions
$\mathcal{R}^+(\mathbb{T})$	The set of positively regressive functions
$\mathbb{T}^z$	Derived form of time scale
$\mathbb{C}(\mathbb{J}, \mathbb{R})$	The Banach space of continuous functions
$P\mathbb{C}(\mathbb{J}, \mathbb{R})$	The Banach space of piecewise continuous functions
Φ	Unknown function
Φ^Δ	Delta derivative of Φ
Φ^σ	$\Phi \circ \sigma$
Ψ	Perturbed function
φ	Nondecreasing function
$\mathrm{r}, \mathrm{q}, s, \eta$	Variables
r_k	Points of impulses

2. Basic Concepts and Remarks

The definitions of TS calculus are recalled from references [29,30]. TS is a concept denoting an arbitrary and non-empty closed subset of the real number line, typically symbolized as $\mathbb{T}$. The forward and backward jump operators with domain and range $\mathbb{T}$ are, respectively, defined as

$$\sigma(s) = \inf\{\mathrm{r} \in \mathbb{T} \colon \mathrm{r} > s\} \text{ and } \rho(s) = \sup\{\mathrm{r} \in \mathbb{T} \colon \mathrm{r} < s\}.$$

The graininess function $\mu \colon \mathbb{T} \to [0, \infty)$ is defined as $\mu(\mathrm{r}) = \sigma(\mathrm{r}) - \mathrm{r}$. The TS's derived form, represented by $\mathbb{T}^z$, is defined as

$$\mathbb{T}^z = \begin{cases} \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{if } \sup \mathbb{T} = \infty. \end{cases}$$

A function denoted as g, mapping from the set $\mathbb{T}$ to the real numbers $\mathbb{R}$, is categorized as rd-continuous if it exhibits continuity at every point where there is a right-dense property within the domain $\mathbb{T}$, and it also has a well-defined left-sided limit at every point with a left-dense property in $\mathbb{T}$. This specific class of functions, possessing both these properties, is denoted as $\mathcal{C}_{rd}(\mathbb{T}, \mathbb{R})$. The notation $\mathcal{C}_{rd}(\mathbb{T} \times \mathbb{R}, \mathbb{R})$ signifies the collection of functions g that are rd-continuous with respect to their first variable and exhibit standard continuity in the second variable when their domain is $\mathbb{T} \times \mathbb{R}$. Furthermore, a function g, again mapping from $\mathbb{T}$ to $\mathbb{R}$, is referred to as regressive (or positively regressive) if the expression $1 + \mu(t)g(t) \neq 0$ (or $1 + \mu(t)g(t) > 0$) holds true for all t belonging to the set $\mathbb{T}^z$. The sets containing rd-continuous regressive functions and rd-continuous positively regressive

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functions are represented as $\mathcal{R}(\mathbb{T})$ and $\mathcal{R}^+(\mathbb{T})$, respectively. The delta derivative as well as the $\Delta-$integral of $f\colon \mathbb{T} \to \mathbb{R}$ are given as

$$f^{\Delta}(\mathrm{r}) = \lim_{s \to \mathrm{r},\, s \neq \sigma(\mathrm{r})} \frac{f(\sigma(\mathrm{r})) - f(s)}{\sigma(\mathrm{r}) - s}, \quad \mathrm{r} \in \mathbb{T}^z$$

and

$$\int_{x_1}^{x_2} f(\mathrm{r})\Delta\mathrm{r} = F(x_2) - F(x_1), \quad x_1, x_2 \in \mathbb{T},$$

where $F^{\Delta} = f$ on $\mathbb{T}^z$.

The solution of equation $\Phi^{\Delta}(\mathrm{r}) = \mathrm{p}(\mathrm{r})\Phi(\mathrm{r})$, $\Phi(\mathrm{r}_0) = \Phi_0$, $\mathrm{r} \in \mathbb{T}^0$, is represented by $e_{\mathrm{p}}(\mathrm{r}, \mathrm{r}_0)$, which is called the exponential function on TS and is defined as follows.

$$e_g(a,b) = \exp\left(\int_a^b \phi_{\mu(s)} g(s)\Delta s\right), \ \ a, b \in \mathbb{T},$$

where

$$\phi_{\mu(\mathrm{r})} g(\mathrm{r}) = \begin{cases} \dfrac{Log(1 + \mu(\mathrm{r})g(\mathrm{r}))}{\mu(\mathrm{r})}, & \text{if } \mu(\mathrm{r}) \neq 0, \\ g(\mathrm{r}), & \text{if } \mu(\mathrm{r}) = 0, \end{cases}$$

is the cylindrical transformation.

For $g, f \in \mathcal{R}(\mathbb{T})$, we define

$$\begin{aligned} (g \oplus f)(\mathrm{r}) &= g(\mathrm{r}) + f(\mathrm{r}) + \mu(\mathrm{r})g(\mathrm{r})f(\mathrm{r}), \\ (\ominus g)(\mathrm{r}) &= -\frac{g(\mathrm{r})}{1 + \mu(\mathrm{r})f(\mathrm{r})}, \\ g \ominus f &= g \oplus (\ominus f). \end{aligned}$$

Let $\mathbb{C}(\mathbb{J}, \mathbb{R})$ represent the Banach space of continuous functions on the interval $\mathbb{J}$ and $P\mathbb{C}(\mathbb{J}, \mathbb{R})$ represent the Banach space of piecewise continuous functions on the same interval with $\|\Phi\| = \sup_{\mathrm{r} \in \mathbb{J}} |\Phi(\mathrm{r})|$. Furthermore, $P\mathbb{C}^1(\mathbb{J}, \mathbb{R}) = \{\Phi \in P\mathbb{C}(\mathbb{J}, \mathbb{R})\colon \Phi^{\Delta} \in P\mathbb{C}(\mathbb{J}, \mathbb{R})\}$ is also a Banach space coupled with the norm $\|\Phi\|_1 = \max\{\|\Phi\|, \|\Phi^{\Delta}\|\}$. Consider the following inequalities:

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$$\begin{cases} \left|\Psi^{\Delta}(\mathrm{r}) + \mathrm{p}(\mathrm{r})\Psi^{\sigma}(\mathrm{r}) - \displaystyle\int_{\mathrm{r}_0}^{\mathrm{r}} f(s, \Psi(s))\Delta s - \int_a^b f(s, \Psi(s))\Delta s\right| \leq \varepsilon, & \mathrm{r} \in \mathbb{J}^z \backslash \{\mathrm{r}_i\}, \\ |\Psi(\mathrm{r}_k^+) - \Psi(\mathrm{r}_k^-) - \Upsilon_k(\Psi(\mathrm{r}_k^-))| \leq \varepsilon, \quad k = \overline{1,\, m}, \end{cases} \tag{2}$$

$$\begin{cases} \left|\Psi^{\Delta}(\mathrm{r}) + \mathrm{p}(\mathrm{r})\Psi^{\sigma}(\mathrm{r}) - \displaystyle\int_{\mathrm{r}_0}^{\mathrm{r}} f(s, \Psi(s))\Delta s - \int_a^b f(s, \Psi(s))\Delta s\right| \leq \varepsilon\varphi(s), & \mathrm{r} \in \mathbb{J}^z \backslash \{\mathrm{r}_i\}, \\ |\Psi(\mathrm{r}_k^+) - \Psi(\mathrm{r}_k^-) - \Upsilon_k(\Psi(\mathrm{r}_k^-))| \leq \varepsilon K, \quad k = \overline{1,\, m}, \end{cases} \tag{3}$$

$$\begin{cases} \left|\Psi^{\Delta}(\mathrm{r}) + \mathrm{p}(\mathrm{r})\Psi^{\sigma}(\mathrm{r}) - \displaystyle\int_{\mathrm{r}_0}^{\mathrm{r}} f(s, \Psi(s))\Delta s - \int_a^b f(s, \Psi(s))\Delta s\right| \leq \varphi(s), & \mathrm{r} \in \mathbb{J}^z \backslash \{\mathrm{r}_i\}, \\ |\Psi(\mathrm{r}_k^+) - \Psi(\mathrm{r}_k^-) - \Upsilon_k(\Psi(\mathrm{r}_k^-))| \leq K, \quad k = \overline{1,\, m}, \end{cases} \tag{4}$$

where $0 < \varepsilon$, $0 \leq K$, $\Psi \in P\mathbb{C}^1(\mathbb{J}, \mathbb{R})$ is a perturbed function, $\varphi \in P\mathbb{C}^1(\mathbb{J}, \mathbb{R}^+)$ is nondecreasing and other terms are the same as in (1).

Definition 1. *The NIVFID adjoint Equation (1) is HU stable on $\mathbb{J}$ if, for each $\Psi \in P\mathbb{C}^1(\mathbb{J}, \mathbb{R})$ satisfying (2), a solution exists $\Phi \in P\mathbb{C}^1(\mathbb{J}, \mathbb{R})$ of (1) such that $|\Phi(\mathrm{r}) - \Psi(\mathrm{r})| \leq C\varepsilon$ for all* $\mathrm{r} \in \mathbb{J}^z \backslash \{\mathrm{r}_i\}, \quad i = \overline{1,\ m}$, *where* $C > 0$.

Definition 2. *The NIVFID adjoint Equation (1) is generalized HU stable on $\mathbb{J}$ if, for each $\Psi \in P\mathbb{C}^1(\mathbb{J}, \mathbb{R})$ satisfying (2), a solution exists $\Phi \in P\mathbb{C}^1(\mathbb{J}, \mathbb{R})$ of (1) with $|\Phi(\mathrm{r}) - \Psi(\mathrm{r})| \leq \gamma(\varepsilon)$ for all* $\mathrm{r} \in \mathbb{J}^z \backslash \{\mathrm{r}_i\}, \quad i = \overline{1,\ m}$, *where* $\gamma \in \mathbb{C}(\mathbb{R}^+, \mathbb{R}^+)$, $\gamma(0) = 0$.

Definition 3. *The NIVFID adjoint Equation (1) is HUR stable with respect to (φ, K) on $\mathbb{J}$ if, for each $\Psi \in P\mathbb{C}^1(\mathbb{J}, \mathbb{R})$ satisfying (3), a solution exists $\Phi \in P\mathbb{C}^1(\mathbb{J}, \mathbb{R})$ of (1) with $|\Phi(\mathrm{r}) - \Psi(\mathrm{r})| \leq C\varepsilon(\varphi(\mathrm{r}) + K)$ for all* $\mathrm{r} \in \mathbb{J}^z \backslash \{\mathrm{r}_i\}, i = \overline{1,\ m}$, *where* $C > 0$.

Definition 4. *The NIVFID adjoint Equation (1) is generalized HUR stable with respect to (φ, K) on $\mathbb{J}$ if, for each $\Psi \in P\mathbb{C}^1(\mathbb{J}, \mathbb{R})$ satisfying (4), a solution exists $\Phi \in P\mathbb{C}^1(\mathbb{J}, \mathbb{R})$ of (1) with $|\Phi(\mathrm{r}) - \Psi(\mathrm{r})| \leq C(\varphi(\mathrm{r}) + K)$ for all* $\mathrm{r} \in \mathbb{J}^z \backslash \{\mathrm{r}_i\}, i = \overline{1,\ m}$, *where* $C > 0$.

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Lemma 1 (Extended integral inequality on TSs (See [28], Theorem 3.1))**.** *Let* $\mathrm{r}_0, \mathrm{r} \in \mathbb{T}$, $\mathrm{r}_0 \leq \mathrm{r}$, $\Phi \in P\mathbb{C}(\mathbb{T}, \mathbb{R})$, $p \in \mathcal{R}(\mathbb{T}^+)$, $a \in P\mathbb{C}(\mathbb{T}, \mathbb{R}^+)$ *be a nondecreasing function and* $c_k \in \mathbb{R}^+$, $k = \overline{1,\ m}$. *Then*

$$\Phi(\mathrm{r}) \leq a(\mathrm{r}) + \int_{\mathrm{r}_0}^{\mathrm{r}} p(\eta)\Phi(\eta)\Delta\eta + \sum_{\mathrm{r}_0 < \mathrm{r}_k < \mathrm{r}} c_k \Phi(\mathrm{r}_k),$$

implies

$$\Phi(\mathrm{r}) \leq a(\mathrm{r}) \prod_{\mathrm{r}_0 < \mathrm{r}_k < \mathrm{r}} (1 + c_k) e_p(\mathrm{r}, \tau), \quad \mathrm{r} \geq \mathrm{r}_0.$$

Remark 1. *From Lemma 2.1 and Remark 2.1 stated in reference [28], it is clear that $\Phi \in P\mathbb{C}^1(\mathbb{J}, \mathbb{R})$ satisfies (1) if and only if*

$$\Phi(\mathrm{r}) = \begin{cases} e_{\ominus \mathrm{p}}(\mathrm{r}, \mathrm{r}_0)\Phi_0 + \displaystyle\sum_{\mathrm{r}_0 < \mathrm{r}_k < \mathrm{r}} e_{\ominus \mathrm{p}}(\mathrm{r}, \mathrm{r}_k) \mathrm{Y}_k(\Phi(\mathrm{r}_k^-)) + \int_{\mathrm{r}_0}^{\mathrm{r}} e_{\ominus \mathrm{p}}(\mathrm{r}, s) \int_{s_0}^{s} f(\mathrm{q}, \Phi(\mathrm{q}))\Delta \mathrm{q} \Delta s \\ + \displaystyle\int_{\mathrm{r}_0}^{\mathrm{r}} e_{\ominus \mathrm{p}}(\mathrm{r}, s) \int_{a}^{b} f(\mathrm{q}, \Phi(\mathrm{q}))\Delta \mathrm{q} \Delta s. \end{cases} \tag{5}$$

Remark 2. *A function $\Psi \in P\mathbb{C}^1(\mathbb{J}, \mathbb{R})$ satisfies (3) if and only if there is $g \in P\mathbb{C}(\mathbb{J}, \mathbb{R})$ and sequence g_k (which is finite) such that $|g(\mathrm{r})| \leq \varepsilon\varphi(\mathrm{r})$ for all $\mathrm{r} \in \mathbb{J}$ and $|g_k| \leq \varepsilon K$ for all $k = \overline{1,\ m}$,*

$$\begin{cases} \Psi^{\Delta}(\mathrm{r}) + \mathrm{p}(\mathrm{r})\Psi^{\sigma}(\mathrm{r}) = \displaystyle\int_{\mathrm{r}_0}^{\mathrm{r}} f(s, \Psi(s))\Delta s + \int_{a}^{b} f(s, \Psi(s))\Delta s + g(\mathrm{r}), \quad \mathrm{r} \in \mathbb{J}^z \backslash \{\mathrm{r}_k\}, \quad k = \overline{1,\ m}, \\ \Psi(\mathrm{r}_k^+) - \Psi(\mathrm{r}_k^-) = \mathrm{Y}_k(\Psi(\mathrm{r}_k^-)) + g_k, \quad k = \overline{1,\ m}, \\ \Psi(\mathrm{r}_0) = \Psi_0 \in \mathbb{R}. \end{cases} \tag{6}$$

Lemma 2. *Every solution $\Psi \in P\mathbb{C}^1(\mathbb{J}, \mathbb{R})$ of (3) also satisfies*

$$\begin{cases} \Bigg| \Psi(\mathrm{r}) - e_{\ominus \mathrm{p}}(\mathrm{r}, \mathrm{r}_0)\Psi_0 - \displaystyle\sum_{\mathrm{r}_0 < \mathrm{r}_k < \mathrm{r}} e_{\ominus \mathrm{p}}(\mathrm{r}, \mathrm{r}_k) \mathrm{Y}_k(\Psi(\mathrm{r}_k^-)) - \int_{\mathrm{r}_0}^{\mathrm{r}} e_{\ominus \mathrm{p}}(\mathrm{r}, s) \int_{s_0}^{s} f(\mathrm{q}, \Psi(\mathrm{q}))\Delta \mathrm{q} \Delta s \\ - \displaystyle\int_{\mathrm{r}_0}^{\mathrm{r}} e_{\ominus \mathrm{p}}(\mathrm{r}, s) \int_{a}^{b} f(\mathrm{q}, \Psi(\mathrm{q}))\Delta \mathrm{q} \Delta s \Bigg| \leq E_k \varepsilon \left(\int_{\mathrm{r}_0}^{\mathrm{r}} \varphi(s)\Delta s + mK \right), \end{cases} \tag{7}$$

where $|e_{\ominus \mathrm{p}}(\mathrm{r}, \cdot)| \leq E_k$, $E_k > 0$, *for* $\mathrm{r} \in (\mathrm{r}_k, \mathrm{r}_{k+1}]_{\mathbb{T}} \subset \mathbb{J}$, $k = \overline{1,\ m}$.

Proof. If (3) is satisfied by $\Psi \in P\mathbb{C}^1(\mathbb{J}, \mathbb{R})$, then the solution of Equation (6), according to Remark 1, is

$$\Psi(\mathrm{r}) = e_{\ominus \mathrm{p}}(\mathrm{r}, \mathrm{r}_0)\Psi_0 + \sum_{\mathrm{r}_0 < \mathrm{r}_k < \mathrm{r}} e_{\ominus \mathrm{p}}(\mathrm{r}, \mathrm{r}_k)(\Upsilon_k(\Psi(\mathrm{r}_k^-)) + g_k) + \int_{\mathrm{r}_0}^{\mathrm{r}} e_{\ominus \mathrm{p}}(\mathrm{r}, s) \int_{s_0}^{s} f(\mathrm{q}, \Psi(\mathrm{q}))\Delta \mathrm{q} \Delta s$$
$$+ \int_{\mathrm{r}_0}^{\mathrm{r}} e_{\ominus \mathrm{p}}(\mathrm{r}, s) \int_{a}^{b} f(\mathrm{q}, \Psi(\mathrm{q}))\Delta \mathrm{q} \Delta s + \int_{\mathrm{r}_0}^{\mathrm{r}} e_{\ominus \mathrm{p}}(\mathrm{r}, s) g(s) \Delta s.$$

Then we may write the following.

$$\begin{aligned}
&\Bigg|\Psi(\mathrm{r}) - e_{\ominus \mathrm{p}}(\mathrm{r}, \mathrm{r}_0)\Psi_0 - \sum_{\mathrm{r}_0 < \mathrm{r}_k < \mathrm{r}} e_{\ominus \mathrm{p}}(\mathrm{r}, \mathrm{r}_k)\Upsilon_k(\Psi(\mathrm{r}_k^-)) - \int_{\mathrm{r}_0}^{\mathrm{r}} e_{\ominus \mathrm{p}}(\mathrm{r}, s) \int_{s_0}^{s} f(\mathrm{q}, \Psi(\mathrm{q}))\Delta \mathrm{q} \Delta s \\
&\quad - \int_{\mathrm{r}_0}^{\mathrm{r}} e_{\ominus \mathrm{p}}(\mathrm{r}, s) \int_{a}^{b} f(\mathrm{q}, \Psi(\mathrm{q}))\Delta \mathrm{q} \Delta s\Bigg| \\
&\leq \int_{\mathrm{r}_0}^{\mathrm{r}} |e_{\ominus \mathrm{p}}(\mathrm{r}, s)||g(s)|\Delta s + \sum_{\mathrm{r}_0 < \mathrm{r}_k < \mathrm{r}} |e_{\ominus \mathrm{p}}(\mathrm{r}, \mathrm{r}_k)||g_k| \\
&\leq \int_{\mathrm{r}_0}^{\mathrm{r}} E_k \varepsilon \varphi(s) \Delta s + \sum_{\mathrm{r}_0 < \mathrm{r}_k < \mathrm{r}} E_k \varepsilon K \\
&\leq \int_{\mathrm{r}_0}^{\mathrm{r}} E_k \varepsilon \varphi(s) \Delta s + m E_k \varepsilon K \\
&= E_k \varepsilon \left(\int_{\mathrm{r}_0}^{\mathrm{r}} \varphi(s) \Delta s + mK \right).
\end{aligned}$$

Hence, (7) is obtained. □

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3. Existence and Uniqueness of Solutions

Here we prove the existence and uniqueness of solutions for (1). We list below the following assumptions:

(C_1) For $f \in \mathcal{C}_{rd}(\mathbb{J} \times \mathbb{R}, \mathbb{R})$, there exists $L > 0$ such that

$$|f(\mathrm{r}, \mathrm{r}_1) - f(\mathrm{r}, \mathrm{r}_2)| \leq L|\mathrm{r}_1 - \mathrm{r}_2|,$$

for all $\mathrm{r} \in \mathbb{J}$ and $\mathrm{r}_i \in \mathbb{R}$, $i \in \{1, 2\}$. Also, there exists $\varrho > 0$ such that $|f(\mathrm{q}, \Phi(\mathrm{q}))| \leq \varrho$.

(C_2) For $\Upsilon_k \colon \mathbb{R} \to \mathbb{R}$, there exists $M_k > 0$ such that

$$|\Upsilon_k(\mathrm{r}_1) - \Upsilon_k(\mathrm{r}_2)| \leq M_k|\mathrm{r}_1 - \mathrm{r}_2|,$$

for all $k = \overline{1, m}$ and $\mathrm{r}_i \in \mathbb{R}$, $i \in \{1, 2\}$. Also, there exists $C_k > 0$ such that $|\Upsilon_k(\Phi(\mathrm{r}_k^-))| \leq C_k$.

(C_3) $\left(\sum_{\mathrm{r}_0 < \mathrm{r}_k < \mathrm{r}} E_k M_k + \frac{\mathrm{r}_f^2}{2} E_k L + \mathrm{r}_f b E_k L \right) < 1.$

(C_4) For a nondecreasing $\varphi \in P\mathbb{C}^1(\mathbb{J}, \mathbb{R})$, there exists $\delta > 0$ such that

$$\int_{\mathrm{r}_0}^{\mathrm{r}} \varphi(\mathrm{q}) \Delta \mathrm{q} \leq \delta \varphi(\mathrm{r}).$$

Theorem 1. *The NIVFID adjoint Equation (1) has a unique solution in $P\mathbb{C}^1(\mathbb{J}, \mathbb{R})$, if ($C_1$)–($C_3$) hold.*

Proof. Let us consider the Banach space $\mathbb{U} := \{\Phi \in P\mathbb{C}^1(\mathbb{J}, \mathbb{R}) \colon |\Phi| \leq \vartheta\}$, where

$$\vartheta := E_k|\Phi_0| + \sum_{\mathrm{r}_0 < \mathrm{r}_k < \mathrm{r}} E_k C_k + E_k \frac{\mathrm{r}_f^2}{2} \varrho + E_k b \mathrm{r}_f \varrho.$$

Define an operator $\Lambda \colon \mathbb{U} \to \mathbb{U}$ by

$$(\Lambda\Phi) := \begin{cases} e_{\ominus \mathrm{p}}(\mathrm{r}, \mathrm{r}_0)\Phi_0 + \sum\limits_{\mathrm{r}_0 < \mathrm{r}_k < \mathrm{r}} e_{\ominus \mathrm{p}}(\mathrm{r}, \mathrm{r}_k)\Upsilon_k(\Phi(\mathrm{r}_k^-)) + \int_{\mathrm{r}_0}^{\mathrm{r}} e_{\ominus \mathrm{p}}(\mathrm{r}, s) \int_{s_0}^{s} f(\mathrm{q}, \Phi(\mathrm{q}))\Delta \mathrm{q}\Delta s \\ + \int_{\mathrm{r}_0}^{\mathrm{r}} e_{\ominus \mathrm{p}}(\mathrm{r}, s) \int_a^b f(\mathrm{q}, \Phi(\mathrm{q}))\Delta \mathrm{q}\Delta s, \quad \mathrm{r} \in (\mathrm{r}_k, \mathrm{r}_{k+1}]_{\mathbb{T}}, \quad k = \overline{1,\ m}. \end{cases} \tag{8}$$

We show that Λ is the Picard operator on $\mathbb{U}$. For this, first we need to show that $\Lambda\colon \mathbb{U} \to \mathbb{U}$. For $\mathrm{r} \in (\mathrm{r}_k, \mathrm{r}_{k+1}]_{\mathbb{T}}$, $k = \overline{1,\ m}$, we have

$$\begin{aligned} |(\Lambda\Phi)(\mathrm{r})| &\le |e_{\ominus \mathrm{p}}(\mathrm{r}, \mathrm{r}_0)||\Phi_0| + \sum_{\mathrm{r}_0 < \mathrm{r}_k < \mathrm{r}} |e_{\ominus \mathrm{p}}(\mathrm{r}, \mathrm{r}_k)||\Upsilon_k(\Phi(\mathrm{r}_k^-))| + \int_{\mathrm{r}_0}^{\mathrm{r}} |e_{\ominus \mathrm{p}}(\mathrm{r}, s)| \int_{s_0}^{s} |f(\mathrm{q}, \Phi(\mathrm{q}))|\Delta \mathrm{q}\Delta s \\ &\quad + \int_{\mathrm{r}_0}^{\mathrm{r}} |e_{\ominus \mathrm{p}}(\mathrm{r}, s)| \int_a^b |f(\mathrm{q}, \Phi(\mathrm{q}))|\Delta \mathrm{q}\Delta s \\ &\overset{(C_1),(C_2)}{\le} E_k|\Phi_0| + \sum_{\mathrm{r}_0 < \mathrm{r}_k < \mathrm{r}} E_k C_k + \int_{\mathrm{r}_0}^{\mathrm{r}} E_k \int_{s_0}^{s} \varrho \Delta \mathrm{q}\Delta s + \int_{\mathrm{r}_0}^{\mathrm{r}} E_k \int_a^b \varrho \Delta \mathrm{q}\Delta s \\ &\le E_k|\Phi_0| + \sum_{\mathrm{r}_0 < \mathrm{r}_k < \mathrm{r}} E_k C_k + \int_{\mathrm{r}_0}^{\mathrm{r}} E_k s \varrho \Delta s + \int_{\mathrm{r}_0}^{\mathrm{r}} E_k b \varrho \Delta s \\ &\le E_k|\Phi_0| + \sum_{\mathrm{r}_0 < \mathrm{r}_k < \mathrm{r}} E_k C_k + E_k \frac{\mathrm{r}^2}{2}\varrho + E_k b \mathrm{r} \varrho \\ &\le E_k|\Phi_0| + \sum_{\mathrm{r}_0 < \mathrm{r}_k < \mathrm{r}} E_k C_k + E_k \frac{\mathrm{r}_f^2}{2}\varrho + E_k b \mathrm{r}_f \varrho \\ &=: \vartheta. \end{aligned}$$

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That is, $\|(\Lambda\Phi)(\mathrm{r})\| \le \vartheta$ and hence $\Lambda\colon \mathbb{U} \to \mathbb{U}$. Next, we see that for, any $\Phi_1, \Phi_2 \in \mathbb{U}$ and $\mathrm{r} \in (\mathrm{r}_k, \mathrm{r}_{k+1}]_{\mathbb{T}}$, $k = \overline{1,\ m}$, a simple calculation yields

$$\begin{aligned} &|(\Lambda\Phi_1)(\mathrm{r}) - (\Lambda\Phi_2)(\mathrm{r})| \\ &\le \sum_{\mathrm{r}_0 < \mathrm{r}_k < \mathrm{r}} |e_{\ominus \mathrm{p}}(\mathrm{r}, \mathrm{r}_k)||\Upsilon_k(\Phi_1(\mathrm{r}_k^-)) - \Upsilon_k(\Phi_2(\mathrm{r}_k^-))| + \int_{\mathrm{r}_0}^{\mathrm{r}} |e_{\ominus \mathrm{p}}(\mathrm{r}, s)| \int_{s_0}^{s} |f(\mathrm{q}, \Phi_1(\mathrm{q})) - f(\mathrm{q}, \Phi_2(\mathrm{q}))|\Delta \mathrm{q}\Delta s \\ &\quad + \int_{\mathrm{r}_0}^{\mathrm{r}} |e_{\ominus \mathrm{p}}(\mathrm{r}, s)| \int_a^b |f(\mathrm{q}, \Phi_1(\mathrm{q})) - f(\mathrm{q}, \Phi_2(\mathrm{q}))|\Delta \mathrm{q}\Delta s \\ &\overset{(C_1),(C_2)}{\le} \sum_{\mathrm{r}_0 < \mathrm{r}_k < \mathrm{r}} E_k M_k |\Phi_1(\mathrm{r}_k^-) - \Phi_2(\mathrm{r}_k^-)| + \int_{\mathrm{r}_0}^{\mathrm{r}} E_k \int_{s_0}^{s} L|\Phi_1(\mathrm{q}) - \Phi_2(\mathrm{q})|\Delta \mathrm{q}\Delta s \\ &\quad + \int_{\mathrm{r}_0}^{\mathrm{r}} E_k \int_a^b L|\Phi_1(\mathrm{q}) - \Phi_2(\mathrm{q})|\Delta \mathrm{q}\Delta s \end{aligned}$$

$$\begin{aligned} &\le \sum_{\mathrm{r}_0 < \mathrm{r}_k < \mathrm{r}} E_k M_k \sup_{\mathrm{r} \in \mathbb{J}} |\Phi_1(\mathrm{r}_k^-) - \Phi_2(\mathrm{r}_k^-)| + \int_{\mathrm{r}_0}^{\mathrm{r}} E_k \int_{s_0}^{s} L \sup_{\mathrm{r} \in \mathbb{J}} |\Phi_1(\mathrm{r}) - \Phi_2(\mathrm{r})|\Delta \mathrm{q}\Delta s \\ &\quad + \int_{\mathrm{r}_0}^{\mathrm{r}} E_k \int_a^b L \sup_{\mathrm{r} \in \mathbb{J}} |\Phi_1(\mathrm{r}) - \Phi_2(\mathrm{r})|\Delta \mathrm{q}\Delta s \\ &\le \sum_{\mathrm{r}_0 < \mathrm{r}_k < \mathrm{r}} E_k M_k |\Phi_1 - \Phi_2| + |\Phi_1 - \Phi_2| \int_{\mathrm{r}_0}^{\mathrm{r}} \int_{s_0}^{s} E_k L \Delta \mathrm{q}\Delta s + |\Phi_1 - \Phi_2| \int_{\mathrm{r}_0}^{\mathrm{r}} \int_a^b E_k L \Delta \mathrm{q}\Delta s \end{aligned}$$

$$\leq \sum_{\mathrm{r}_0<\mathrm{r}_k<\mathrm{r}} E_k M_k |\Phi_1 - \Phi_2| + |\Phi_1 - \Phi_2| \int_{\mathrm{r}_0}^{\mathrm{r}} s E_k L \Delta s + |\Phi_1 - \Phi_2| \int_{\mathrm{r}_0}^{\mathrm{r}} b E_k L \Delta s$$
$$\leq \sum_{\mathrm{r}_0<\mathrm{r}_k<\mathrm{r}} E_k M_k |\Phi_1 - \Phi_2| + |\Phi_1 - \Phi_2| \frac{\mathrm{r}^2}{2} E_k L + |\Phi_1 - \Phi_2| \mathrm{r} b E_k L$$
$$\leq \sum_{\mathrm{r}_0<\mathrm{r}_k<\mathrm{r}} E_k M_k |\Phi_1 - \Phi_2| + |\Phi_1 - \Phi_2| \frac{\mathrm{r}_f^2}{2} E_k L + |\Phi_1 - \Phi_2| \mathrm{r}_f b E_k L$$
$$\leq |\Phi_1 - \Phi_2| \left(\sum_{\mathrm{r}_0<\mathrm{r}_k<\mathrm{r}} E_k M_k + \frac{\mathrm{r}_f^2}{2} E_k L + \mathrm{r}_f b E_k L \right).$$

In the view of (C_3), we see that Λ is contractive and, hence, is a Picard operator on $\mathbb{U}$. Therefore, Λ has a unique fixed point in $\mathbb{U}$ which is the unique solution of (1) (from (8)) in $P\mathbb{C}^1(\mathbb{J}, \mathbb{R})$. □

4. Ulam-Type Stability Results

Now, we investigate the Ulam-type stability results for model (1).

Theorem 2. *The NIVFID adjoint Equation (1) has HUR stability with respect to* (φ, K) *on* $\mathbb{J}$*, if* (C_1)–(C_4) *hold.*

Proof. Let $\Psi \in P\mathbb{C}^1(\mathbb{J}, \mathbb{R})$ satisfy (3) and $\Phi \in P\mathbb{C}^1(\mathbb{J}, \mathbb{R})$ be a solution of (1) satisfying $\Phi_0 = \Psi_0$. Then, in the view of Remark 1, we can write

$$\Phi(\mathrm{r}) = \begin{cases} e_{\ominus \mathrm{p}}(\mathrm{r}, \mathrm{r}_0)\Psi_0 + \sum_{\mathrm{r}_0<\mathrm{r}_k<\mathrm{r}} e_{\ominus \mathrm{p}}(\mathrm{r}, \mathrm{r}_k) \Upsilon_k(\Phi(\mathrm{r}_k^-)) + \int_{\mathrm{r}_0}^{\mathrm{r}} e_{\ominus \mathrm{p}}(\mathrm{r}, s) \int_{s_0}^{s} f(\mathrm{q}, \Phi(\mathrm{q})) \Delta \mathrm{q} \Delta s \\ + \int_{\mathrm{r}_0}^{\mathrm{r}} e_{\ominus \mathrm{p}}(\mathrm{r}, s) \int_a^b f(\mathrm{q}, \Phi(\mathrm{q})) \Delta \mathrm{q} \Delta s, \quad \mathrm{r} \in (\mathrm{r}_k, \mathrm{r}_{k+1}]_{\mathbb{T}}, \quad k = \overline{1, m}. \end{cases}$$

Now, since $\Psi \in P\mathbb{C}^1(\mathbb{J}, \mathbb{R})$ satisfies (3), in the view of Remark 2, we can write

$$\begin{cases} \Psi^{\Delta}(\mathrm{r}) + \mathrm{p}(\mathrm{r}) \Psi^{\sigma}(\mathrm{r}) = \int_{\mathrm{r}_0}^{\mathrm{r}} f(s, \Psi(s)) \Delta s + \int_a^b f(s, \Psi(s)) \Delta s + g(\mathrm{r}), \quad \mathrm{r} \in \mathbb{J}^{z} \backslash \{\mathrm{r}_k\}, \\ \Psi(\mathrm{r}_k^+) - \Psi(\mathrm{r}_k^-) = \Upsilon_k(\Psi(\mathrm{r}_k^-)) + g_k, \end{cases}$$

where $|g(\mathrm{r})| \leq \varepsilon \varphi(\mathrm{r})$ for all $\mathrm{r} \in \mathbb{J}$ and $|g_k| \leq \varepsilon K$ for all $k = \overline{1, m}$. Thus

$$\Psi(\mathrm{r}) = e_{\ominus \mathrm{p}}(\mathrm{r}, \mathrm{r}_0)\Psi_0 + \sum_{\mathrm{r}_0<\mathrm{r}_k<\mathrm{r}} e_{\ominus \mathrm{p}}(\mathrm{r}, \mathrm{r}_k)(\Upsilon_k(\Psi(\mathrm{r}_k^-)) + g_k) + \int_{\mathrm{r}_0}^{\mathrm{r}} e_{\ominus \mathrm{p}}(\mathrm{r}, s) \int_{s_0}^{s} f(\mathrm{q}, \Psi(\mathrm{q})) \Delta \mathrm{q} \Delta s$$
$$+ \int_{\mathrm{r}_0}^{\mathrm{r}} e_{\ominus \mathrm{p}}(\mathrm{r}, s) \int_a^b f(\mathrm{q}, \Psi(\mathrm{q})) \Delta \mathrm{q} \Delta s + \int_{\mathrm{r}_0}^{\mathrm{r}} e_{\ominus \mathrm{p}}(\mathrm{r}, s) g(s) \Delta s.$$

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Using Lemma 2, we can write for $\mathrm{r} \in (\mathrm{r}_k, \mathrm{r}_{k+1}]_{\mathbb{T}}$, $k = \overline{1,\ m}$,

$$
\begin{aligned}
&|\Psi(\mathrm{r}) - \Phi(\mathrm{r})| \\
&= \bigg| \Psi(\mathrm{r}) - e_{\ominus\mathrm{p}}(\mathrm{r}, \mathrm{r}_0)\Psi_0 - \sum_{\mathrm{r}_0<\mathrm{r}_k<\mathrm{r}} e_{\ominus\mathrm{p}}(\mathrm{r}, \mathrm{r}_k)\Upsilon_k(\Psi(\mathrm{r}_k^-)) - \int_{\mathrm{r}_0}^{\mathrm{r}} e_{\ominus\mathrm{p}}(\mathrm{r}, s) \int_{s_0}^{s} f(\mathrm{q}, \Psi(\mathrm{q}))\Delta\mathrm{q}\Delta s \\
&\quad - \int_{\mathrm{r}_0}^{\mathrm{r}} e_{\ominus\mathrm{p}}(\mathrm{r}, s) \int_{a}^{b} f(\mathrm{q}, \Psi(\mathrm{q}))\Delta\mathrm{q}\Delta s \bigg| + \sum_{\mathrm{r}_0<\mathrm{r}_k<\mathrm{r}} |e_{\ominus\mathrm{p}}(\mathrm{r}, \mathrm{r}_k)(\Upsilon_k(\Psi(\mathrm{r}_k^-)) - \Upsilon_k(\Phi(\mathrm{r}_k^-)))| \\
&\quad + \left| \int_{\mathrm{r}_0}^{\mathrm{r}} e_{\ominus\mathrm{p}}(\mathrm{r}, s) \int_{s_0}^{s} \Big(f(\mathrm{q}, \Psi(\mathrm{q})) - f(\mathrm{q}, \Phi(\mathrm{q})) \Big) \Delta\mathrm{q}\Delta s \right| \\
&\quad + \left| \int_{\mathrm{r}_0}^{\mathrm{r}} e_{\ominus\mathrm{p}}(\mathrm{r}, s) \int_{a}^{b} \Big(f(\mathrm{q}, \Psi(\mathrm{q})) - f(\mathrm{q}, \Phi(\mathrm{q})) \Big) \Delta\mathrm{q}\Delta s \right| \\
&\leq E_k\varepsilon\left(\int_{\mathrm{r}_0}^{\mathrm{r}} \varphi(s)\Delta s + mK \right) + \sum_{\mathrm{r}_0<\mathrm{r}_k<\mathrm{r}} M_k E_k |\Psi(\mathrm{r}_k^-) - \Phi(\mathrm{r}_k^-)| \\
&\quad + \int_{\mathrm{r}_0}^{\mathrm{r}} |e_{\ominus\mathrm{p}}(\mathrm{r}, s)| \int_{s_0}^{s} |(f(\mathrm{q}, \Psi(\mathrm{q})) - f(\mathrm{q}, \Phi(\mathrm{q})))| \Delta\mathrm{q}\Delta s \\
&\quad + \int_{\mathrm{r}_0}^{\mathrm{r}} |e_{\ominus\mathrm{p}}(\mathrm{r}, s)| \int_{a}^{b} |(f(\mathrm{q}, \Psi(\mathrm{q})) - f(\mathrm{q}, \Phi(\mathrm{q})))| \Delta\mathrm{q}\Delta s \\
&\overset{(\mathrm{C}_1),(\mathrm{C}_2)}{\leq} E_k\varepsilon(\delta\varphi(\mathrm{r}) + mK) + \sum_{\mathrm{r}_0<\mathrm{r}_k<\mathrm{r}} M_k E_k |\Psi(\mathrm{r}_k^-) - \Phi(\mathrm{r}_k^-)| \\
&\quad + \int_{\mathrm{r}_0}^{\mathrm{r}} E_k \int_{s_0}^{s} L|\Psi(\mathrm{q}) - \Phi(\mathrm{q})|\Delta\mathrm{q}\Delta s + \int_{\mathrm{r}_0}^{\mathrm{r}} E_k \int_{a}^{b} L|\Psi(\mathrm{q}) - \Phi(\mathrm{q})|\Delta\mathrm{q}\Delta s \\
&\leq E_k\varepsilon(\delta\varphi(\mathrm{r}) + mK) + \sum_{\mathrm{r}_0<\mathrm{r}_k<\mathrm{r}} M_k E_k |\Psi(\mathrm{r}_k^-) - \Phi(\mathrm{r}_k^-)| \\
&\quad + \int_{\mathrm{r}_0}^{\mathrm{r}} E_k L \left(\int_{s_0}^{s} + \int_{a}^{b} \right) |\Psi(\mathrm{q}) - \Phi(\mathrm{q})|\Delta\mathrm{q}\Delta s.
\end{aligned}
$$

According to Lemma 1, we can write for all $\mathrm{r} \in (\mathrm{r}_m, \mathrm{r}_{m+1}]_{\mathbb{T}}$,

$$
\begin{aligned}
|\Psi(\mathrm{r}) - \Phi(\mathrm{r})| &\leq E_k\varepsilon(\delta\varphi(\mathrm{r}) + mK) \prod_{\mathrm{r}_0<\mathrm{r}_k<\mathrm{r}} (1 + M_k E_k) e_{\mathbb{P}}(\mathrm{r}, \mathrm{r}_0) \\
&\leq E_k\varepsilon(\delta + m)(\varphi(\mathrm{r}) + K) \prod_{\mathrm{r}_0<\mathrm{r}_k<\mathrm{r}} (1 + M_k E_k) e_{\mathbb{P}}(\mathrm{r}, \mathrm{r}_0),
\end{aligned}
$$

where $\mathbb{P} := E_k L\left(\int_{s_0}^{s} + \int_{a}^{b} \right)\Delta\mathrm{q}$ is a positively regressive bounded function. The property of the exponential function $e_{\mathbb{P}}(\mathrm{r}, \mathrm{r}_0) \leq e^{\mathbb{P}(\mathrm{r}-\mathrm{r}_0)}$ allows us to write

$$
|\Psi(\mathrm{r}) - \Phi(\mathrm{r})| \leq E_k\varepsilon(\delta + m)(\varphi(\mathrm{r}) + K) \prod_{\mathrm{r}_0<\mathrm{r}_k<\mathrm{r}} (1 + M_k C_k) e^{\mathbb{P}(\mathrm{r}-\mathrm{r}_0)}.
$$

This yields

$$
|\Psi(\mathrm{r}) - \Phi(\mathrm{r})| \leq E_k\varepsilon(\delta + m)(\varphi(\mathrm{r}) + K) \prod_{\mathrm{r}_0<\mathrm{r}_k<\mathrm{r}} (1 + M_k C_k) e^{\mathbb{P}^*(\mathrm{r}_f-\mathrm{r}_0)},
$$

where $\mathbb{P} \leq \mathbb{P}^*$ and $\mathbb{P}^*$ is a fixed value. The desired assertion now follows by choosing $C := E_k(\delta + m)\prod_{\mathrm{r}_0<\mathrm{r}_k<\mathrm{r}}(1 + M_k E_k)e^{\mathbb{P}^*(\mathrm{r}_f-\mathrm{r}_0)}$. □

Based on Theorem 2, we can obtain the following corollaries.

Corollary 1. *The NIVFID adjoint Equation (1) has the generalized HUR stability on* $\mathbb{J}$, *if* (C_1)–(C_4) *hold.*

Proof. In the proof of Theorem 2, set ε to be equal to 1, and the proof can be completed accordingly. □

Corollary 2. *The NIVFID adjoint Equation (1) has the HU stability on* $\mathbb{J}$, *if* (C_1)–(C_4) *hold.*

Proof. Take $\varphi(r) = K = 1$ in the proof of Theorem 2 and we obtain

$$|\Psi(r) - \Phi(r)| \leq 2E_k \varepsilon (r_f - r_0 + m) \prod_{r_0 < r_k < r} (1 + M_k E_k) e^{\mathbb{P}^*(r_f - r_0)},$$

which is the required result. □

Corollary 3. *The NIVFID adjoint Equation (1) has the generalized HU stability on* $\mathbb{J}$, *if* (C_1)–(C_4) *hold.*

Proof. Take $\gamma(\epsilon) = 2E_k \epsilon (r_f - r_0 + m) \prod_{r_0 < r_k < r} (1 + M_k E_k) e^{\mathbb{P}^*(r_f - r_0)}$ and the result follows from the Corollary 2. □

5. Example

Let $\mathbb{T}$ be an arbitrary TS which do not contain integers and consider the following NIVFID adjoint equation

$$\begin{cases} \Phi^{\Delta}(r) + \dfrac{1}{r-2}\Phi^{\sigma}(r) = \displaystyle\int_{r_0}^{r} e^{-12}(s + \sin(\Phi(s)))\Delta s + \int_0^4 e^{-12}(s + \sin(\Phi(s)))\Delta s, \\ \Phi(0) = 1, \quad r \in [0,4]_{\mathbb{T}} \backslash \{2\}, \\ Y_1(\Phi(2^-)) = \Phi(2^+) - \Phi(2^-) = \dfrac{2 + \sin(\Phi(2^-))}{5} \end{cases} \tag{9}$$

and its associated inequality

$$\begin{aligned} &\left| \Psi^{\Delta}(r) + \frac{1}{r-2}\Psi^{\sigma}(r) - \int_{r_0}^{r} e^{-12}(s + \sin(\Psi(s)))\Delta s - \int_0^4 e^{-12}(s + \sin(\Psi(s)))\Delta s \right| \leq \varepsilon, \ r \in [0,4]_{\mathbb{T}} \backslash \{2\}, \\ &|Y_1(\Psi(2^-)) - \Psi(2^+) + \Psi(2^-)| \leq \varepsilon, \quad k = 1. \end{aligned} \tag{10}$$

Let $\mathbb{J} = [0,4]_{\mathbb{T}} \backslash \{2\}$, $r_0 = 0$, $r_f = 4$, $p(r) = \frac{1}{r-2}$, and $f(r, \Phi(r)) = r + \sin(\Phi(r))$ for $r \in \mathbb{J} = [0,4]_{\mathbb{T}} \backslash \{2\}$. If $\Psi \in P\mathbb{C}^1([0,4]_{\mathbb{T}}, \mathbb{R})$ satisfies (10), then there exists $g \in P\mathbb{C}^1([0,4]_{\mathbb{T}}, \mathbb{R})$ and $g_0 \in \mathbb{R}$ such that $|g(r)| \leq \varepsilon$ for $r \in \mathbb{J} = [0,4]_{\mathbb{T}} \backslash \{2\}$ and $|g_0| \leq \varepsilon$. Thus

$$\begin{aligned} &\Psi^{\Delta}(r) + \frac{1}{r-2}\Psi^{\sigma}(r) = \int_{r_0}^{r} e^{-12}(s + \sin(\Psi(s)))\Delta s + \int_0^4 e^{-12}(s + \sin(\Psi(s)))\Delta s + g(r), \ r \in \mathbb{J} = [0,4]_{\mathbb{T}} \backslash \{2\}, \\ &\Psi(0) = 1, \\ &Y_1(\Psi(2^-)) = \Psi(2^+) - \Psi(2^-) + g_0, \end{aligned}$$

and the solution of Equation (9) is

$$\begin{aligned} \Phi(r) = {} & e_{\ominus p}(r,0) + e_{\ominus p}(r,2^-)Y_1(\Phi(2^-)) + \int_{r_0}^{r} e^{-12} e_{\ominus p}(r,s) \int_{r_0}^{s} (u + \sin(\Phi(u)))\Delta u \Delta s \\ & + \int_{r_0}^{r} e^{-12} e_{\ominus p}(r,s) \int_0^4 (u + \sin(\Phi(u)))\Delta u \Delta s. \end{aligned}$$

We see that the function $f(r, \Phi(r)) = r + \sin(\Phi(r))$ satisfies (C_1). The condition (C_2) is also satisfied, since $|Y_1(\Phi(2^-)) - Y_1(\Psi(2^-))| \leq \frac{1}{5}|\Phi(2^-) - \Psi(2^-)|$. Further, for $p = \frac{1}{r-2}$, $e_{\ominus p}(r,s)$ is also bounded because $|e_{\ominus p}(r,s)| \leq 1$. The simple calculations yield

$$|\Lambda(\Phi) - \Lambda(\Psi)| \leq \left(\frac{1}{5} + 8e^{-12} + 16e^{-12} \right) |\Phi - \Psi|.$$

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Obviously, $\left(\frac{1}{5} + 8e^{-12} + 16e^{-12}\right) < 1$, so ($C_3$) holds for Equation (9). Hence, (C_1)–(C_3) hold for Equation (9). Therefore, according to Theorem 1, Equation (9) has a unique solution in $P\mathbb{C}^1([0,4]_{\mathbb{T}}, \mathbb{R})$. Moreover, if we take $\varphi(\mathrm{r}) = e^{\mathrm{r}}$, which is a continuous and increasing function with $\int_{\mathrm{r}_0}^{\mathrm{r}} e^s \Delta s \leq \int_{\mathrm{r}_0}^{\mathrm{r}} e^s ds = e^{\mathrm{r}} - e^{\mathrm{r}_0} \leq e^{\mathrm{r}} < 10e^{\mathrm{r}}$ for $\delta = 10$, i.e., $\int_{\mathrm{r}_0}^{\mathrm{r}} e^s \Delta s < 10e^{\mathrm{r}}$. Thus, ($C_4$) holds. Further, we see that $|\Phi(\mathrm{r}) - \Psi(\mathrm{r})| \leq \varphi(\mathrm{r})$, $C = 1$, $\varphi = e^{(\cdot)}$ and $K = 0$. Hence, according to Theorem 2, Equation (9) is HUR stable with respect to $(e^{(\cdot)}, 0)$ on $\mathbb{J} = [0,4]_{\mathbb{T}} \backslash \{2\}$.

6. Conclusions

Our study examined the Ulam stability of NIVFID adjoint equations within the context of finite TS intervals. We have derived our findings through the effective application of an extended integral inequality designed for TSs. The Ulam stability plays a pivotal role in approximating solutions to problems when exact solutions are challenging to obtain. Impulsive dynamic equations are well-suited for representing the complex behavior of systems that exhibit a blend of continuous and discrete characteristics, occasionally manifesting unexpected discontinuous transitions as they evolve. As a consequence, the findings outlined in this paper are of significant relevance in diverse academic domains, including approximation theory, control theory, optimization and their associated fields of application. We foresee the seamless extension of these outcomes to encompass systems governed by impulsive dynamic equations and to the broader context of Banach spaces. Moreover, a promising avenue for future research lies in the exploration of impulsive problems (1) that incorporate appropriate time delays, offering intriguing opportunities for further investigation. Also, Equation (1) is useful in the mathematical modeling of several real-life phenomena in which abrupt changes occur during the process. The results obtained in the paper are generalizable for the NIVFID adjoint equations with delay, NIVFID adjoint equations with nonlocal conditions and for NIVFID adjoint systems on Banach spaces.

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