Journal of Nonlinear and Convex Analysis
Volume 24, Number 9, 2023, 1967–1976

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL


# APPLICATIONS OF DIFFERENTIAL TRANSFORM AND BELL POLYNOMIALS TO VARIOUS TYPES OF DIFFERENTIAL EQUATIONS

GIRIRAJ METHI, ANIL KUMAR, RUPAL AGGARWAL, AND SANKET TIKARE

Abstract. In this paper, a new technique using differential transform and Bell polynomials is proposed and applied to some different types of nonlinear delay differential equations (NDDEs) and fractional differential equations. Differential transform converts the problem into recursive relation and Bell polynomials deal with various forms of nonlinearity. The presented technique can be considered a simple and efficient tool for solving NDDEs. The proposed technique has been investigated on three concrete problems. The absolute error and maximum absolute errors are computed to show the efficiency and reliability of the presented method.

## 1. Introduction

Differential transform method (DTM) has been introduced by Pukhov [15] and Zhou [23] as 'Taylor transform' and applied it to the study of electrical circuits. DTM has close relation with Taylor expansion of real analytic functions. It has applications in solving different types of problems for all classes of differential equations (ordinary, partial, delayed, fractional, fuzzy etc.). The recent developments and applications of DTM are discussed in [1,9,11,12,17,20,21] and references therein.

DTM is an iterative method for determining the series solution of both linear and nonlinear differential equations. In comparison to the traditional series method, which requires symbolic computation, DTM transforms the differential equation into algebraic equations that can be solved recursively.

Many series solution methods such as Adomian decomposition method, Variational iteration method, Homotopy perturbation method, and including differential transform method are difficult to apply on problems containing nonlinear terms. Hence, it is essential to find a technique to deal with problems involving nonlinear terms. We propose application of Bell polynomials to handle nonlinearity. Bell polynomials were first used to examine set partitions. Bell polynomials appear in a variety of applications, including combinatorics, analysis, statistics, and so on [7].

In the present paper, application of differential transform together with Bell polynomials is studied on nonlinear delay and fractional differential equations. The error analysis is discussed in detail.

The paper is organized as follows. In Section 2, we introduce the main idea and basic formulae of the differential transformation and provide necessary results for the nonlinearities involving partial ordinary Bell polynomials. Applications of the

---

2020 *Mathematics Subject Classification.* 34K07, 34K37, 65L03.
*Key words and phrases.* Delay differential equations, fractional differential equation, differential transform, Bell polynomials, convergence, error estimate.

proposed method are presented in Section 3. The conclusion is discussed in Section 4.

## 2. Description of the method

In this section, we discuss the main idea and basic formulae of the differential transformation as well as notations and results related to the transformation of general nonlinear terms.

2.1. **Idea of differential transform.** Let $w$ be a real analytical function in a domain $\Omega$ and $v_0$ be an arbitrary point in $\Omega$. Then, $w$ can be expanded in a Taylor series in a neighbourhood of the point $v_0$ [16].

**Definition 2.1.** The differential transform of k$^{\text{th}}$ derivative of the function $w$ at $v_0$ is defined as

$$W(k)[v_0] := \frac{1}{\Gamma(\alpha k+1)}\left[\frac{d^{\alpha k}w(v)}{dv^{\alpha k}}\right]_{v=v_0}, \tag{2.1}$$

where $0<\alpha\leq 1$ and $W(k)[v_0]$ represents differential transform of $w(v)$ at $v=v_0$.

**Definition 2.2.** The inverse differential transformation is given by

$$w(v)=\sum_{k=0}^{\infty} W(k)[v_0](v-v_0)^{\alpha k}. \tag{2.2}$$

Using definitions 2.1 and 2.2, function $w$ can be represented in the form of Taylor series:

$$w(v)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha k+1)}\left[\frac{d^{\alpha k}w(v)}{dv^{\alpha k}}\right]_{v=v_0}(v-v_0)^{\alpha k}. \tag{2.3}$$

The solution is approximated by

$$w(v)=\sum_{k=0}^{\infty} W(k)[v_0](v-v_0)^{\alpha k}. \tag{2.4}$$

2.2. **Faà di Bruno formula and Bell polynomials.** Some necessary notations and definitions of Bell polynomials are given below.

**Lemma 2.3** ([9])**.** *The partial ordinary Bell polynomials $\widehat{B}_{k,l}(\widehat{x}_1,\ldots,\widehat{x}_{k-l+1})$, $l=0,1,2,\ldots$, $k\geq l$ satisfy the recurrence relation*

$$\widehat{B}_{k,l}(\widehat{x}_1,\ldots,\widehat{x}_{k-l+1})=\sum_{i=1}^{k-l+1}\frac{il}{k}\widehat{x}_i\widehat{B}_{k-i,l-1}(\widehat{x}_1,\ldots,\widehat{x}_{k-i-l+2}), \tag{2.5}$$

*where $\widehat{B}_{0,0}=1$ and $\widehat{B}_{k,0}=0$ for $k\geq 1$.*

**Theorem 2.4** ([9])**.** *Assuming $g$ and $f$ to be analytic functions near $v_0$ and $g(v_0)$ respectively, and $h$ is the composition defined as $h(v)=(f\circ g)(v)=f(g(v))$ for all $v\in\mathbb{R}$. Let the differential transform of functions $g$, $f$, and $h$ respectively, at $v_0$, $g(v_0)$, and $v_0$ are denoted by $G(k)$, $F(k)$, and $H(k)$ respectively. Then, $H(k)$ satisfy the recursive relation*

$$H(0) = F(0),$$

$$H(k) = \sum_{l=1}^{k} F(l)\widehat{B}_{k,l}(G(1),\ldots,G(k-l+1)) \text{ for } k \geq 1, \tag{2.6}$$

*where $\widehat{B}_{k,l}$ are Bell polynomials as in Lemma 2.3.*

**Theorem 2.5.** *Assuming $g$ and $f$ to be analytic functions near $v_0$ and $g(v_0)$ respectively, and $h$ is the composite defined as $h(v) = (f \circ g)(v) = f\left(g(\frac{v}{\beta})\right)$ for all $v \in \mathbb{R}$ and $\beta \neq 0$. Let the differential transform of functions $g$, $f$, and $h$ respectively at $v_0$, $g(v_0)$, and $v_0$ are denoted by $G(k)$, $F(k)$, and $H(k)$ respectively. Then, $H(k)$ satisfy the relations*

$$H(0) = F(0),$$

$$H(k) = \left(\frac{1}{\beta}\right)^k \sum_{l=1}^{k} F(l)\widehat{B}_{k,l}(G(1),\ldots,G(k-l+1)) \text{ for } k \geq 1, \tag{2.7}$$

*where $\widehat{B}_{k,l}$ are Bell polynomials as in Lemma 2.3.*

2.3. **Error estimate.** For comparison [13], absolute error and maximum absolute error are computed and defined as

$$E_N(v) := |w(v) - w_N(v)|,$$
$$E_{N,\infty} := \max_{0 \leq v \leq 1} E_N(v),$$

where $w$ is the analytical solution and $w_N$ is the truncated series solution with degree $N$. Furthermore, the relative error between exact and approximate solution is defined by

$$R_N(v) := \frac{E_N(v)}{|w(v)|}.$$

## 3. Applications

In this section, three examples are discussed to show accuracy and robustness of the presented method. The MATHEMATICA software version 11 has been used for numerical computations.

**Example 3.1.** Consider the following nonlinear proportional DDE

$$w''(v) = 2w'(v) - w\left(\frac{v}{2}\right) + \ln\left(w\left(\frac{v}{4}\right)\right) + \sqrt{w(v)} - \frac{v}{2}, \quad 0 \leq v \leq 1 \tag{3.1}$$

with initial conditions

$$w(0) = 1 \text{ and } w'(0) = 2. \tag{3.2}$$

The exact solution of (3.1)–(3.2) is given by

$$w(v) = e^{2v}. \tag{3.3}$$

For $g_1(v) = w(\frac{v}{4})$ and $f_1(x) = \ln(x)$, we write $h_1(v) = f_1(g_1(v))$. Also, for $g_2(v) = w(v)$ and $f_2(x) = \sqrt{x}$, we write $h_2(v) = f_2(g_2(v))$. Let $F_1(k)$ be the differential transform of $f_1(x)$. Then, using Definition 2.1, we get

$$F_1(k) = \frac{(-1)^{k+1}}{k}, \quad k \geq 1. \tag{3.4}$$

Let $H_1(k)$ be the differential transform of $h_1(v)$. Then, using Theorem 2.5, we obtain

$$
\begin{aligned}
H_1(0) &= 0, \\
H_1(k) &= \left(\frac{1}{4}^k\right) \sum_{l=1}^{k} F_1(l)\widehat{B}_{k,l}\left(W(1),\ldots,W(k-l+1)\right) \text{ for } k \geq 1.
\end{aligned} \tag{3.5}
$$

Let $F_2(k)$ be the Differential transform of $f_2(x)$. Then, using Definition 2.1, we get

$$F_2(k) = \frac{n(n-1)\ldots(n-k+1)}{k!}, \text{ where } n = \frac{1}{2}, \tag{3.6}$$

Let $H_2(k)$ be the differential transform of $h_2(v)$. Then, using Theorem 2.4, we obtain

$$
\begin{aligned}
H_2(0) &= 1, \\
H_2(k) &= \sum_{l=1}^{k} F_2(l)\widehat{B}_{k,l}\left(W(1),\ldots,W(k-l+1)\right) \text{ for } k \geq 1.
\end{aligned} \tag{3.7}
$$

Now, applying differential transform to (3.1)–(3.2), we obtain the following recursive relation

$$
\begin{aligned}
W(k+2) = \frac{1}{(k+1)(k+2)} \bigg( & 2(K+1)W(k+1) - \frac{1}{2^k}W(k) + H_1(k) \\
& + H_2(k) - \frac{1}{2}\delta(k-1) \bigg),
\end{aligned} \tag{3.8}
$$

$$W(0) = 1 \text{ and } W(1) = 2. \tag{3.9}$$

Solving equations (3.4)–(3.9), we obtain different components. Now, with the help of Equation (2.4), the series solution is given by

$$w(v) = 1 + 2v + 2v^2 + \frac{4}{3}v^3 + \ldots, \tag{3.10}$$

which converges to the exact solution given by (3.3).

TABLE 1. Comparison of numerical solution $w$ with the exact solution when $N = 12$ for Example 1

| $v$ | $w(v)$ | $w_N(v)$ | $R_N(v)$ |
|---|---|---|---|
| 0.1 | 1.221402758 | 1.221402758 | 0 |
| 0.2 | 1.491824698 | 1.491824698 | 2.4E-14 |
| 0.3 | 1.822118800 | 1.822118800 | 2.6E-12 |
| 0.4 | 2.225540928 | 2.225540928 | 6.8E-11 |
| 0.5 | 2.718281828 | 2.718281826 | 8.3E-10 |
| 0.6 | 3.320116923 | 3.320116902 | 6.1E-09 |
| 0.7 | 4.055199967 | 4.055199834 | 3.2E-08 |
| 0.8 | 4.953032424 | 4.953031755 | 1.3E-07 |
| 0.9 | 6.049647464 | 6.049644666 | 4.6E-07 |
| 1.0 | 7.389056099 | 7.389046016 | 1.3E-06 |

TABLE 2. Maximum absolute errors for $w$ of Example 1

| $N$ | $E_{N,\infty}$ |
|---|---|
| 5 | 3.8E-01 |
| 10 | 3.4E-04 |
| 15 | 2.8E-08 |

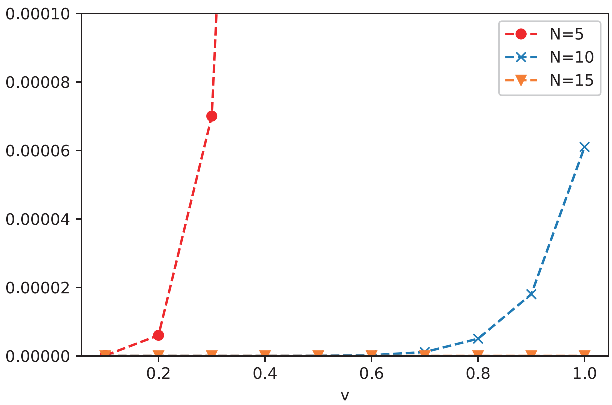


FIGURE 1. Absolute errors for $w$ when N=5, 10, and 15 of Example 1

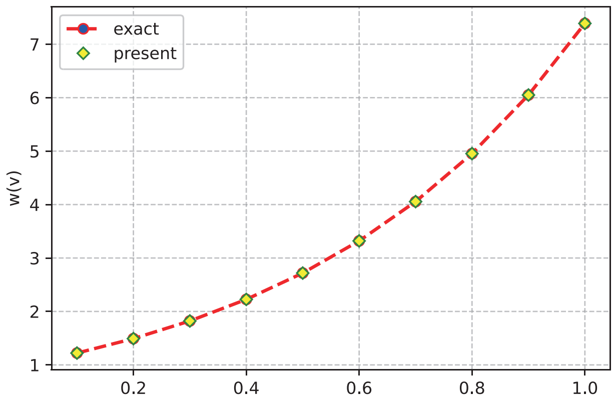


FIGURE 2. Comparison of the exact solution and present solution (for N=12) for Example 1.

**Example 3.2.** Consider the following nonlinear fractional differential equation

$$D^{\alpha}w(v) = 2w(v) + 4w(v)\ln w(v), \quad \alpha \in (1,2), \tag{3.11}$$

with initial conditions

$$w(0) = 1 \text{ and } w'(0) = 0. \tag{3.12}$$

Denote $h(v) = f(g(v))$, where $g(v) = w(v)$ and $f(x) = \ln(x)$. Let $F(k)$ be the differential transform of $f(x)$. Then, using Definition 2.1, we get

$$F(k) = \frac{(-1)^{k+1}}{k}, \quad k \geq 1. \tag{3.13}$$

Let $H(k)$ be the differential transform of $h(v)$. Then, using Theorem 2.4, we obtain

$$\begin{aligned} H(0) &= 0, \\ H(k) &= \sum_{l=1}^{k} F(l)\widehat{B}_{k,l}(W(1),\dots,W(k-l+1)) \text{ for } k \geq 1. \end{aligned} \tag{3.14}$$

Now, applying differential transform to equations (3.11)–(3.12), we obtain the following recursive relation

$$\frac{\Gamma\left(\alpha\left(\frac{k}{2}+1\right)+1\right)}{\Gamma\left(\alpha\frac{k}{2}+1\right)}W(k+2) = 2W(k) + 4\sum_{r=0}^{k} W(r)H(k-r), \tag{3.15}$$

$$W(0) = 1 \quad \text{and } W(1) = 0. \tag{3.16}$$

Solving equation (3.13)–(3.16), we obtain different component. Now, with the help of Equation (2.4), the series solution for $\alpha = 2$ is given by

$$w(v) = 1 + v^2 + \frac{1}{2}v^4 + \frac{1}{6}v^6 + \dots, \tag{3.17}$$

which converges to the exact solution given by

$$w(v) = e^{v^2}. \tag{3.18}$$

TABLE 3. Comparison of numerical solution $w$ with different values of $\alpha$ for Example 2

| $v$ | $w_N(v)$ for $\alpha$=2 | $w_N(v)$ for $\alpha$=1.99 | $w_N(v)$ for $\alpha = 1.95$ | $w_N(v)$ for $\alpha = 1.90$ | $w_N(v)$ for $\alpha = 1.85$ |
|---|---|---|---|---|---|
| 0.1 | 1.010050167 | 1.010144209 | 1.010525278 | 1.011012331 | 1.011510739 |
| 0.2 | 1.040810774 | 1.041205798 | 1.042810939 | 1.044873477 | 1.046997849 |
| 0.3 | 1.094174284 | 1.095137146 | 1.099067149 | 1.104160431 | 1.109461171 |
| 0.4 | 1.173510871 | 1.175420158 | 1.183259688 | 1.193536138 | 1.204379579 |
| 0.5 | 1.284025416 | 1.287443822 | 1.301582247 | 1.320374366 | 1.340537904 |
| 0.6 | 1.433329407 | 1.439111099 | 1.463226635 | 1.495799037 | 1.531432718 |
| 0.7 | 1.632316133 | 1.641771686 | 1.681588142 | 1.736351593 | 1.79759037 |
| 0.8 | 1.896480128 | 1.911635342 | 1.976131092 | 2.066647508 | 2.170379997 |
| 0.9 | 2.247902941 | 2.271911016 | 2.375280172 | 2.523628315 | 2.698335611 |
| 1.0 | 2.718253968 | 2.756059233 | 2.920930112 | 3.163425691 | 3.457747953 |

**Example 3.3.** Consider the following nonlinear fractional differential equation

$$D^{\alpha}w(v) + e^{w(v)} = 0, \quad \alpha \in (0,1), \tag{3.19}$$

with initial condition

$$w(0) = 0. \tag{3.20}$$

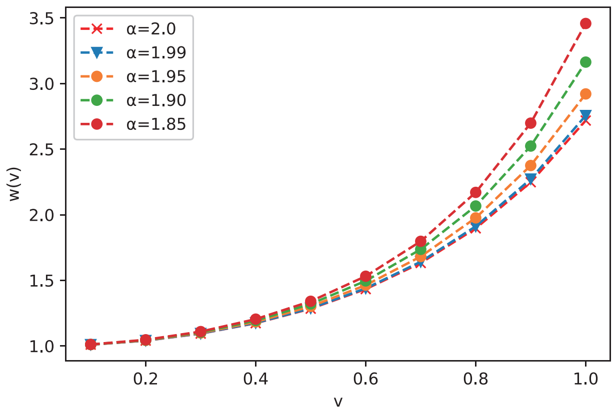


FIGURE 3. Comparison for different values of $\alpha$ for Example 2

Denote $h(v) = f(g(v))$, where $g(v) = w(v)$ and $f(x) = e^x$. Let $F(k)$ be the differential transform of $f(x)$. Then, this $F$ is given by

$$F(k) = \frac{1}{k!}. \tag{3.21}$$

Let $H(k)$ be the differential transform of $h(v)$. Then, using Theorem 2.4, we obtain

$$\begin{aligned} H(0) &= 1, \\ H(k) &= \sum_{l=1}^{k} F(l)\widehat{B}_{k,l}(W(1), \ldots, W(k-l+1)) \text{ for } k \geq 1. \end{aligned} \tag{3.22}$$

Now, applying differential transform to equations (3.19)–(3.20), we obtain the following recursive relation

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} W(k+1) = -H(k) \text{ and} \tag{3.23}$$

$$W(0) = 0. \tag{3.24}$$

Solving equations (3.21)–(3.23), we obtain different component. Now, with the help of equation (2.4), the series solution for $\alpha = 1$ is given by

$$w(v) = -v + \frac{v^2}{2} - \frac{v^3}{3} + \frac{v^4}{4} - \frac{v^5}{5} + \ldots, \tag{3.25}$$

which converges to the exact solution given by

$$w(v) = -1 - \ln(v). \tag{3.26}$$

TABLE 4. Comparison of numerical solution $w$ with different values of $\alpha$ for Example 3

| $v$ | $w_N(v)$ for $\alpha$=1 | $w_N(v)$ for $\alpha$=0.9 | $w_N(v)$ for $\alpha = 0.8$ | $w_N(v)$ for $\alpha = 0.7$ | $w_N(v)$ for $\alpha = 0.6$ |
|---|---|---|---|---|---|
| 0.1 | -0.095310333 | -0.098510668 | -0.100930158 | -0.102723108 | -0.103739694 |
| 0.2 | -0.182330666 | -0.188494575 | -0.190887687 | -0.193200417 | -0.194111748 |
| 0.3 | -0.262461000 | -0.274656890 | -0.272324251 | -0.274604035 | -0.275160766 |
| 0.4 | -0.336981333 | -0.363518996 | -0.347766892 | -0.350524314 | -0.351990711 |
| 0.5 | -0.407291666 | -0.463853617 | -0.420579417 | -0.426249460 | -0.432817976 |
| 0.6 | -0.475152000 | -0.587119947 | -0.495723787 | -0.510045147 | -0.531024469 |
| 0.7 | -0.542922333 | -0.747898783 | -0.580521500 | -0.614434125 | -0.667210693 |
| 0.8 | -0.613802666 | -0.964327650 | -0.685414982 | -0.757475839 | -0.871248821 |
| 0.9 | -0.692073000 | -1.258535935 | -0.824728973 | -0.964046035 | -1.184335782 |
| 1.0 | -0.783333333 | -1.657080012 | -1.017431915 | -1.267116376 | -1.661046335 |

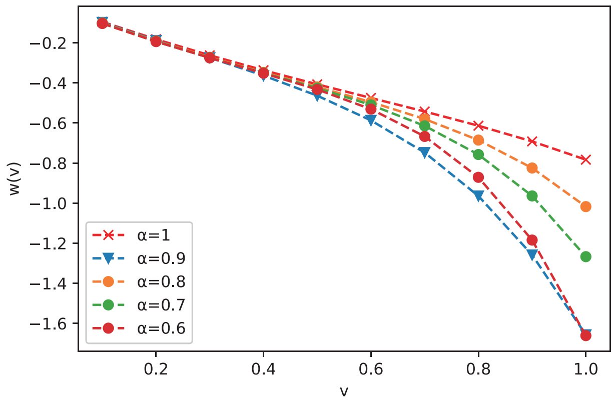


FIGURE 4. Comparison for different values of $\alpha$ for Example 3

## 4. CONCLUSION

We have obtained approximate solutions for different types of nonlinear proportional delay differential equations and fractional differential equations with initial conditions using differential transform and Bell polynomials. The solution is calculated in the form of a convergent power series with easily computable components. It is also worth pointing out that the advantages of the present method in place of numerical methods are simplicity, robustness, and small size of calculation. The present method is a very good tool to handle different types of nonlinearity easily. Moreover, its small errors made it a powerful technique. Therefore, the present method can be seen as a promising tool for solving different types of nonlinear differential equations.

## ACKNOWLEDGMENTS

The authors are thankful to the referees for their valuable comments and suggestions towards improvement of the paper.

## References


[1] M. Abdulkawi, *Solution of Cauchy type singular integral equations of the first kind by using differential transform method*, Applied Mathematical Modelling **39** (2015), 2107–2118.

[2] T. Akkaya, S. Yalcinbas and M. Sezer, *Numeric solutions for the pantograph type delay differential equation using first Boubaker polynomials*, Appl. Math. Comput. **219** (2013), 9484–9492.

[3] B. Benhammouda, H. V. Leal and L. H. Martinez, *Procedure for exact solutions of nonlinear pantograph delay differential equations*, Br. J. Math. Comput. Sci. **4** (2014), 2738–2751.

[4] J. Biazar and B. Ghanbari, *The homotopy perturbation method for solving neutral functional-differential equations with proportional delays*, J. King Saud Univ. Sci. **24** (2010), 33–37.

[5] L. Blanco-Cocom, A. G. Estrella and E. Avila-Vales, *Solving delay differential systems with history functions by the Adomian decomposition method*, Appl. Math. Comput. **218** (2013), 5994–6011.

[6] X. Chen and L. Wang, *The variational iteration method for solving a neutral functional-differential equation with proportional delays*, Comput. Math. Appl. **59** (2010), 2696–2702.

[7] L. Comtet, *Advanced Combinatorics-The Art of Finite and Infinite Expansions*, Springer Dordrecht, Holland, 1974.

[8] H. Jafari, M. Mahmoudi and M. Z. Noori Skandari, *A new numerical method of solve pantograph delay differential equations with convergence analysis*, Adv. Differ. Equ. **2021** (2021): Article number 129.

[9] A. Kumar, G. Methi and S. Tikare, *Applications of differential transform method on some functional differential equations*, in: Advances in Mathematical Analysis and its Applications, Chapman and Hall/CRC, Boca Raton, 2022, pp. 169–190.

[10] A. Kurnaz and G. Oturanc, *The Differential transform approximation for the system ordinary differential equations*, Int. J. Comput. Math. **82** (2005), 709–719.

[11] G. Methi, *Reduced differential transform method for exact solution of parabolic and hyperbolic partial differential equations*, J. Appl. Comput. Sci. Math. **11** (2017), 25–27.

[12] G. Methi and A. Kumar, *Numerical solution of linear and higher-order delay differential equations using the coded differential transform method*, Comput. Res. Model. **11** (2020), 1091–1099.

[13] G. Methi, A. Kumar and J. Rebenda, *Applications of the differential transformation to three-point singular boundary value problems for ordinary differential equations*, Int. J. Appl. Math. Comput. Sci. **29**, (2022), 73–89.

[14] S. R. M. Noori and N. Taghizadeh, *Modified differential transform method for solving linear and nonlinear pantograph type of differential and Volterra integro-differential equations with proportional delays*, Adv. Differ. Equ. **2020** (2020): Article number 649.

[15] G. E. Pukhov, *Differential transforms and circuit theory*, Int. J. Circuit Theory Appl. **10** (1982), 265–276.

[16] Z. Odibat, S. Momani and V. Erturk, *Generalized differential transform method: application to differential equations of fractional order*, Appl. Math. Comput. **197** (2008), 467–477.

[17] A. S. V. Ravi Kanth and K. Aruna, *Solution of singular two-point boundary value problems using differential transformation method*, Phys. Lett. A **372** (2008), 4671–4673.

[18] J. Rebenda, *An application of Bell polynomials in numerical solving of nonlinear differential equations*, in: APLIMAT Proceeding, 2018, pp. 1–10.

[19] J. Rebenda and Z. Šmarda, *Numerical algorithm for nonlinear delayed differential systems of nth order*, Adv. Differ. Equ. **2019** (2019): Article number 26.

[20] J. Rebenda and Z. Pátíková, *Differential transform algorithm for functional differential equations with time-dependent delays*, Complexity **2020** (2020): Article ID 2854574.

[21] U. Sarp, F. Evirgen and S. Ikikardes, *Applications of differential transform method to solve systems of ordinary and partial differential equations*, Balíkesir Üniv. Fen Bilim. Enst. derg. (Online) **20** (2018), 135–156.

[22] Z. H. Yu, *Variational iteration method for solving the multi-pantograph delay equation*, Phys. Lett. A **372** (2008), 6475–6479.

[23] J. K. Zhou, *Differential Transformation and Its Applications for Electrical Circuits*, Huazhong University Press, Wuhan, China, 1986.

*Manuscript received October 14, 2022*
*revised March 4, 2023*

G. Methi
Department of Mathematics & Statistics, Manipal University Jaipur, Rajasthan 303 007, India
*E-mail address*: `giriraјmethi@gmail.com`

A. Kumar
Department of Mathematics & Statistics, Manipal University Jaipur, Rajasthan 303 007, India
*E-mail address*: `anil.digit@gmail.com`

R. Aggarwal
Department of Mathematics & Statistics, Manipal University Jaipur, Rajasthan 303 007, India
*E-mail address*: `rupal.3ma@gmail.com`

S. Tikare
Department of Mathematics, Ramniranjan Jhunjhunwala College, Mumbai, Maharashtra 400 086, India
*E-mail address*: `sankettikare@rjcollege.edu.in`