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Chaplygin's method for second-order neutral differential equations with piecewise constant deviating arguments

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Abstract

The purpose of this paper is to extend Chaplygin's theorem to second-order neutral differential equations with piecewise constant delay. We start with some auxiliary results concerning upper and lower solutions of second-order neutral differential equations. We then use these extended results to find bounds in terms of Chaplygin sequences for the solution of the addressed problem. These bounds, formed by the construction of upper and lower solutions, are shown to converge to the unique solution of the equation. Finally, we show that the error estimates obtained are sharper than those for ordinary and first-order neutral differential equations.

Keywords Chaplygin's sequence · Neutral differential equations · Lower solution · Upper solution

Mathematics Subject Classification 34K40 · 34K05

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1 Introduction

Delay differential equations have been gaining a lot of interest owing to their widespread applications in modeling several physical phenomenon, particularly those appearing in biological systems. These equations exhibiting hybrid properties-discrete and continuous, have been exhaustively studied, see [2, 8, 11, 15, 25], to mention a few. Cooke and Wiener [7] extended these concepts, coupled with comparison principles and the monotone iterative technique, to differential equations with piecewise constant delay of generalized-type. Guyker [10] derived the existence criteria for the eventually periodic solutions of this class of differential equations. In 2016, Marzban and Hoseini [22] developed useful computational techniques to efficiently solve these equations. Later, Muminov [23] proposed a method that reduces a $2n$ periodic solvable equation to a system of $n + 1$ linear equations. Their method is primarily to obtain periodical solutions of second-order neutral differential equations with piecewise constant arguments. Cabada and Ferreiro [5] studied first-order neutral differential equations with piecewise constant arguments coupled with nonlinear boundary conditions. Chiu [6] obtained several qualitative results for piecewise constant linear and nonlinear delay differential equations with impulsive effects in addition to proving a variation of parameters formula with Green function-type and Gronwall-type integral inequality.

Several methods like the method of quasilinearization [3] provide a monotone sequence of approximations. These approximations are shown to converge to the unique solution of the given nonlinear differential equation. In order to get better results, these techniques were refined in [19] by loosening restrictions on the nonlinear function. For nonlinear differential equations, the monotone iterative technique was developed by Ladde and Lakshmikantham [17]. A variation of the successive iteration approach that includes constructing a series of functions that approximately get closer to the original solution was developed by Chaplygin in 1954. This method is known as the Chaplygin method. Lakshmikantham and Leela [18] employed this method to study nonlinear ordinary differential equations. Later, in 1980, Kamout [13] extended this method to first-order functional partial differential equations. Zhukovskaya and Filippova [26] studied first-order delay differential equations using Chaplygin's method. Further, Kumari and Valaulikar [16] employed Chaplygin's method to study first-order neutral differential equations. The Chaplygin method has been gaining momentum and has been applied to stochastic differential equations by Soheili and Amini [24]. As studies on time scales has become very popular since the end of the twentieth century, Jin and Zhang [12] derived a generalized Chaplygin formula for nonholonomic systems, and studied the Nöether theorem for generalized Chaplygin system on time scales. In 2020, Zhukovskaya and Serova [27] obtained the existence theorem and an estimate for the solution of a two-point boundary value problem for an implicit differential equation with a deviating argument, which is similar to the Chaplygin theorem on differential inequalities. Very recently, Benarab [4] gave an interesting application of Chaplygin's method for solvability of the Cauchy problem of implicit differential equations of order n in the construction of the estimates of the solution.

Motivated by these facts, in this paper, we follow an approach more rigorous in analysis and slightly different from [16] to extend Chaplygin's method to second-order neutral differential equations with piecewise constant deviating arguments of the form

$$y''(t) = g(t, y(t), y'(t), y([t]), y'([t]), y''([t])), \quad t \in [0, \infty) \tag{1}$$

with the initial conditions

$$y(0) = \alpha_0, \; y'(0) = \alpha_1, \; y''(0) = \alpha_2, \tag{2}$$

where $\alpha_i \in \mathbb{R}$, $i = 0, 1, 2$, $[\cdot]$ denotes the greatest integer function, and $g : E \to \mathbb{R}$, $E \subseteq \mathbb{R}^+ \times \mathbb{R}^5$ is a bounded function satisfying the following conditions.

1. g is twice continuously differentiable.
2. g possess all second-order partial derivatives which are positive and sufficiently small.

We shall show that the results obtained in this paper are better than those existing in literature, and particularly in [16].

The rest of the paper is organized as follows. The following section provides all necessary definitions, examples, lemmas, and extensions of existing results to second-order neutral differential equations with piecewise constant deviating arguments. In Sect. 3, we prove Chaplygin's theorem using the Mean Value Theorem and establish the error analysis which ascertains that the results in this paper are better than those existing in literature. Section 4 summarizes the results obtained. We conclude by giving future scope to motivate interested researchers.

2 Essential preliminaries

In this portion, we review several fundamental results that will come in handy later on. The set of all real-valued continuous functions defined on I is denoted by $C(I, \mathbb{R})$. Throughout this manuscript, E will denote an open set on $\mathbb{R}^+ \times \mathbb{R}^5$.

Definition 1 A function $y \in C([0, \infty), \mathbb{R})$ is said to be a solution of (1)–(2) if

(i) derivatives y' and y'' exist at each $t \in [0, \infty)$ except possibly at $[t] \in [0, \infty)$ (where only one-sided derivatives exist).
(ii) Equation (1) holds on each interval $[n, n+1) \subset [0, \infty)$, $n \in \mathbb{N}$.
(iii) the conditions (2) hold.

Definition 2 Let $p \in C([0, \infty), \mathbb{R})$. We define the Dini derivatives, used in this paper, as:

$$\mathcal{D}_{+}p(t) = \limsup_{h\to 0^+} \frac{p(t+h)-p(t)}{h} = \liminf_{h\to 0^+} \frac{p(t+h)-p(t)}{h},$$

and with $\mathcal{D}_{+}p(t) = q(t)$,

$$\mathcal{D}_{2,+}p(t) = \limsup_{h\to 0^+} \frac{q(t+h)-q(t)}{h} = \liminf_{h\to 0^+} \frac{q(t+h)-q(t)}{h}.$$

Zygmund's lemma that will be used is stated below.

Lemma 1 (*Zygmund's Lemma* [18, *Lemma* 1.2.1]) *Let* $u \in \mathrm{C}([0,\beta),\mathbb{R})$, $\beta \in \mathbb{R}^+$ *be such that* $\mathcal{D}u(t) < 0$ *for* $t \in [0,\beta)$, $\mathcal{D}$ *being a fixed Dini derivative. Then,* u *is nonincreasing on* $[0,\beta)$. *Further, if* $v, w \in \mathrm{C}([0,\beta),\mathbb{R})$, *and for some fixed Dini derivative* $\mathcal{D}$, $\mathcal{D}v(t) \le w(t)$ *for* $t \in [0,\beta)$. *Then,* $\mathcal{D}_{2,+}v(t) \le w(t)$ *for* $t \in [0,\beta)$.

Lemma 2 [18, *Lemma* 1.3.1] *Let* $g \in \mathrm{C}(E,\mathbb{R})$ *and suppose that* $[0,\beta)$ *is the largest interval on which the maximal solution* ζ *of* (1)–(2) *exists. Suppose* $[0,\tau]$ *is a compact subinterval of* $[0,\beta)$. *Then, there is an* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$, *the maximal solution* $\zeta(t,\varepsilon)$ *of* (1)–(2) *exists on* $[0,\tau]$ *and*

$$\lim_{\varepsilon\to 0} \zeta(t,\varepsilon) = \zeta(t)$$

uniformly on $[0,\tau]$.

We now turn to define upper and lower solutions of the associated neutral differential equation.

Definition 3 (Lower solution) Let $p \in \mathrm{C}([0,\beta],\mathbb{R})$, $\beta \in \mathbb{R}^+$ be such that $\mathcal{D}_{+}p(t), \mathcal{D}_{2,+}p(t)$ exist for $t \in [0,\beta]$ and $(t, p(t), p'(t), p([t]), p'([t]), p''([t])) \in E$. We say that p is a lower solution of (1)–(2) if it satisfies the differential inequalities

$$\mathcal{D}_{2,+}p(t) \le g(t, p(t), p'(t), p([t]), p'([t]), p''([t])), \quad t \in [0,\beta]$$

and

$$p(0) \le \alpha_0,\ p'(0) \le \alpha_1,\ p''(0) \le \alpha_2.$$

Definition 4 (Upper solution) Let $q \in \mathrm{C}([0,\beta],\mathbb{R})$, $\beta \in \mathbb{R}^+$ be such that $\mathcal{D}_{+}q(t), \mathcal{D}_{2,+}q(t)$ exist for $t \in [0,\beta]$ and $(t, q(t), q'(t), q([t]), q'([t]), q''([t])) \in E$. We say that q is an upper solution of (1)–(2) if it satisfies the differential inequalities

$$\mathcal{D}_{2,+}q(t) \ge g(t, q(t), q'(t), q([t]), q'([t]), q''([t])), \quad t \in [0,\beta]$$

and

$$q(0) \ge \alpha_0,\ q'(0) \ge \alpha_1,\ q''(0) \ge \alpha_2.$$

Since Eq. (1) is relatively less studied, we give an example illustrating the upper and lower solution.

Example 1 Consider the following second-order neutral differential equation

$$y''(t) = y(t)\sin(y''([t])) + y'(t) + \cos(y([t])), \quad t \in [0, \infty), \tag{3}$$

with the initial conditions

$$y(0) = y'(0) = y''(0) = 1. \tag{4}$$

Putting $y(t) = e^t$ in the right side of (3), we get

$$e^t \sin(e^{[t]}) + e^t + \cos(e^{[t]}) \le e^t + e^t + 1 = 2e^t + 1.$$

Also, $y''(t) = e^t \le 2e^t + 1$. Thus, $y(t) = e^t$ is an upper solution of (3)–(4). Now, putting $y(t) = e^{-t}$ in the right side of (3), we get

$$e^{-t}\sin(e^{[-t]}) - e^{-t} + \cos(e^{[-t]}) \ge -e^{-t} - e^{-t} - 1 = -2e^{-t} - 1.$$

Also, $y''(t) = e^{-t} \ge -2e^{-t} - 1$. Thus, $y(t) = e^{-t}$ is a lower solution of (3)–(4).

The Arzelà–Ascoli theorem that will be used is stated below.

Theorem 1 (*See* [14, *Theorem* 8.26]) *Let I be a closed and bounded interval in $\mathbb{R}$ and $\{f_n\}$ be a sequence of functions that is uniformly bounded and equicontinuous on I. Then there is a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ that converges uniformly on I.*

We shall start by proving the following lemma which will be repeatedly used in establishing our main results.

Lemma 3 *Let $g \in C(E, \mathbb{R})$ be such that $g(t,x,y,z,u,v)$ is strictly increasing in both z and u for $(t, x, y, v) \in [0, \beta) \times \mathbb{R}^3$, $\beta \in \mathbb{R}^+$. Assume that*

$$\begin{aligned} g(t, x_1, y_1, z_1, u_1, v_1) - g(t, x_2, y_2, z_2, u_2, v_2) \le{}& L_1(x_1 - x_2) + L_2(y_1 - y_2) \\ &+ L_3(z_1 - z_2) + L_4(u_1 - u_2) \\ &+ L_5(v_1 - v_2), \end{aligned}$$

where $x_1 \ge x_2$, $y_1 \ge y_2$, $z_1 \ge z_2$, $u_1 \ge u_2$, $v_1 \ge v_2$ and L_i, $i \in \{1, 2, 3, 4, 5\}$ are positive constants with $L := \max\{L_1, L_2, L_3, L_4, L_5\}$ and $L_5 \le \frac{3L-2}{9L}$. Suppose that p and q are, respectively, lower and upper solutions of (1)–(2) satisfying the following conditions.

(C_1) For $t \in [0, \beta)$, $\beta \in \mathbb{R}^+$,

$$(t, p(t), p'(t), p([t]), p'([t]), p''([t])), (t, q(t), q'(t), q([t]), q'([t]), q''([t])) \in E.$$

(C_1) $p(0) \le y(0) \le q(0), p'(0) \le y'(0) \le q'(0)$, and $p''(0) \le y''(0) \le q''(0)$, where y is the unique solution of (1)–(2).

Then $p(t) \le y(t) \le q(t)$ for all $t \in [0, \beta)$, $\beta \in \mathbb{R}^+$.

Proof Let p_n and q_n denote, respectively, lower and upper solutions of (1)–(2) in $[n, n+1)$, $n \in \mathbb{N}_0$ such that $\{p_n\}$ and $\{q_n\}$ converge uniformly on $[0, \beta)$ to p and q respectively. We shall prove $p_n(t) \leq y_n(t) \leq q_n(t)$, $t \in [n, n+1)$, $n \in \mathbb{N}_0$, where $\{y_n\}$ is a sequence of functions converging uniformly on $[0, \beta)$ to y, which is the unique solution of (1)–(2). For this, we shall only show $y_n(t) \leq q_n(t)$ for $t \in [n, n+1)$ and the proof for $p_n(t) \leq y_n(t)$, on the same interval is virtually identical. Assume that $g(t, q_n(t), q_n'(t), q_n([t]), q_n'([t]), q_n''([t])) < q_n''(t)$. This gives for $t \in [n, n+1)$

$$g(t, q_n(t), q_n'(t), q_n(n), q_n'(n), q_n''(n)) < q_n''(t).$$

<u>Claim</u>: $y_n(t) < q_n(t)$ for $t \in [n, n+1)$.

If this is false, then there would exists $n_1 \in [n, n+1)$ such that

$$q_n(n_1) = y_n(n_1), \quad y_n(t) < q_n(t) \quad \text{for } t \in (n, n_1), \tag{5}$$

and for $t \in (n_1, n+1)$

$$y_n(t) > q_n(t).$$

Choose $h > 0$ small enough so that $n + h < n_1$. Then,

$$q_n(n+h) = q_n(n) + hq_n'(n) \text{ and } y_n(n+h) = y_n(n) + hy_n'(n).$$

Now, from Eq. (5), we obtain $q_n(n+h) - y_n(n+h) > 0$.
Then $q_n(n) + hq_n'(n) - y_n(n) - hy_n'(n) > 0$ which gives $h(q_n'(n) - y_n'(n)) > 0$.

That is, $q_n'(n) > y_n'(n)$. Since $h > 0$, $y_n'(n_1 - h) < q_n'(n_1 - h)$, we have $q_n'(n_1) - q_n'(n_1 - h) < y_n'(n_1) - y_n'(n_1 - h)$, which results in,

$$q_n''(n_1) = \lim_{h \to 0} \frac{q_n'(n_1) - q_n'(n_1 - h)}{h} < \lim_{h \to 0} \frac{y_n'(n_1) - y_n'(n_1 - h)}{h} = y_n''(n_1).$$

That is,

$$q_n''(n_1) < y_n''(n_1). \tag{6}$$

Since g is strictly increasing and $y_n(n_1) = q_n(n_1)$ and $y_n'(n) < q_n'(n)$, it follows that

$$y_n''(n_1) \leq q_n''(n_1), \tag{7}$$

which contradicts (6). Hence $y_n(t) < q_n(t)$ for $t \in [n, n+1)$

Next, we define $w_n(t) := y_n(t) - \varepsilon e^{3Lt}$ for $t \in [n, n+1)$, where $\varepsilon > 0$ such that ε tends to 0 whenever n tending to ∞. Then $w_n'(t) = y_n'(t) - 3L\varepsilon e^{3Lt}$. This gives

$$w_n(0) = y_n(0) - \varepsilon < y_n(0) \overset{(C_2)}{\leq} q_n(0)$$

and

$$w_n'(0) = y_n'(0) - 3L\varepsilon < y_n'(0) \overset{(C_2)}{\leq} q_n'(0).$$

Further, for $t \in [n, n+1)$

$$\begin{aligned}
w_n''(t) &= y_n''(t) - 9L^2\varepsilon e^{3Lt}\\
&\overset{(1)}{=} g(t, y_n(t), y_n'(t), y_n([t]), y_n'([t]), y_n''([t])) - 9L^2\epsilon e^{3Lt}\\
&= g(t, y_n(t), y_n'(t), y_n([t]), y_n'([t]), y_n''([t])) - g(t, w_n(t), w_n'(t), w_n([t]), w_n'([t]), w_n''([t]))\\
&\quad + g(t, w_n(t), w_n'(t), w_n([t]), w_n'([t]), w_n''([t])) - 9L^2\varepsilon e^{3Lt}\\
&\leq L_1(y_n(t) - w_n(t)) + L_2(y_n'(t) - w_n'(t)) + L_3(y_n([t]) - w_n([t]))\\
&\quad + L_4(y_n'([t]) - w_n'([t])) + L_5(y_n''([t]) - w_n''([t]))\\
&\quad + g(t, w_n(t), w_n'(t), w_n([t]), w_n'([t]), w_n''([t])) - 9L^2\varepsilon e^{3Lt}\\
&\leq L_1\varepsilon e^{3Lt} + L_2 3L\varepsilon e^{3Lt} + L_3\varepsilon e^{3Ln} + L_4 3L\varepsilon e^{3Ln} + L_5 9L^2\varepsilon e^{3Ln}\\
&\quad + g(t, w_n(t), w_n'(t), w_n([t]), w_n'([t]), w_n''([t])) - 9L^2\varepsilon e^{3Lt}\\
&\leq L\varepsilon e^{3Lt} + 3L^2\varepsilon e^{3Lt} + L\varepsilon e^{3Ln} + +3L^2\varepsilon e^{3Ln} + (3L-2)L\varepsilon e^{3Ln}\\
&\quad + g(t, w_n(t), w_n'(t), w_n([t]), w_n'([t]), w_n''([t])) - 9L^2\varepsilon e^{3Lt}\\
&= L\varepsilon e^{3Lt}(1 - 6L) + L\varepsilon e^{3Ln}(6L - 1) + g(t, w_n(t), w_n'(t), w_n([t]), w_n'([t]), w_n''([t]))\\
&\leq L\varepsilon e^{\max\{3Lt, 3Ln\}}(1 - 6L + 6L - 1) + g(t, w_n(t), w_n'(t), w_n([t]), w_n'([t]), w_n''([t])).
\end{aligned}$$

Thus, $w_n''(t) \leq g(t, w_n(t), w_n'(t), w_n([t]), w_n'([t]), w_n''([t])$ for $t \in [n, n+1)$.
Since for $t \in [n, n+1)$, $g(t, q_n(t), q_n'(t), q_n([t]), q_n'([t]), q_n''([t])) \leq q_n''(t)$ and $w_n(n) < q_n(n) w_n'(n) < q_n'(n) w_n''(n) < q_n''(n)$, we obtain $w_n(t) < q_n(t)$ for all $t \in [n, n+1)$. Now, letting ε tends to 0, we get

$$y_n(t) \leq q_n(t) \text{ for all } t \in [n, n+1).$$

On similar lines, it is easy to see that

$$p_n(t) \leq y_n(t) \text{ for all } t \in [n, n+1).$$

As our desired result is obtained, the proof is now complete. □

The following comparison theorem will be used to prove the sharpness of our results.

Theorem 2 *Let* $g \in \mathrm{C}(E, \mathbb{R})$ *and suppose that* $[0, \beta)$ *is the largest interval on which the maximal solution* ζ *of* (1)–(2) *exists. Let* $w \in \mathrm{C}([0, \beta, \mathbb{R})$ *be such that* $(t, w(t), w'(t), w([t]), w'([t]), w''([t])) \in E$ *for* $t \in [0, \beta)$, $w_0 \leq \alpha_0$, $w_0' \leq \alpha_1$, $w_0'' \leq \alpha_2$, *and for a fixed Dini derivative* $\mathcal{D}_2$,

$$\mathcal{D}_2 w(t) \leq g(t, w(t), w'(t), w([t]), w'([t]), w''([t])), \quad t \in [0, \beta). \tag{8}$$

Then

$$w(t) \leq \zeta(t), \quad t \in [0, \beta). \tag{9}$$

Proof In view of Lemma 1, we can rewrite Eq. (8) as follows.

$$\mathcal{D}_{2,+}w(t) \leq g(t, w(t), w'(t), w([t]), w'([t]), w''([t])), \quad t \in [0, \beta). \tag{10}$$

Let $0 < \tau < \beta$. By Lemma 2, the maximal solution $\zeta(t, \epsilon)$ of (10) exists on $[0, \tau]$ for all $\varepsilon > 0$ sufficiently small, and

$$\zeta(t) = \lim_{\varepsilon \to 0} \zeta(t, \varepsilon), \tag{11}$$

uniformly on $[0, \tau]$. Now, consider the following second-order neutral differential equation

$$y''(t) = g(t, y(t), y'(t), y([t]), y'([t]), y''([t])) + \varepsilon, \tag{12}$$

with an initial condition

$$y(0) = \alpha_0 + \varepsilon, \quad y'(0) = \alpha_1 + \varepsilon, \quad y''(0) = \alpha_2 + \varepsilon. \tag{13}$$

Using (12)–(13) and (10), and in view of Lemma 3, we obtain

$$w(t) < \zeta(t, \varepsilon), \quad t \in [0, \tau].$$

In view of (11), this last inequality proves the required assertion. This completes the proof. □

3 Main results

We shall now prove Chaplygin's theorem for (1)–(2).

Theorem 3 *Let $g : E \to \mathbb{R}$ be a continuous function, where*

$$E := \Big\{(t, x, y, z, u, v) \in \mathbb{R}^+ \times \mathbb{R}^5 : |y(t) - \alpha_0| \leq a,\ |y'(t) - \alpha_1| \leq b,\ |y([t]) - \alpha_0| \leq c,$$
$$|y'([t]) - \alpha_1| \leq d,\ |y''([t]) - \alpha_2| \leq e\Big\},$$

for some suitable nonnegative real constants a,b,c,d,e. Suppose that there exists $M > 0$ such that $g(t, x, y, z, u, v) \leq M$ for all $(t, x, y, z, u, v) \in E$. Further, assume that all first-order partial derivatives of g, $D_i g$, $i = 2, 3, 4, 5, 6$ exist and second-order partial derivatives of g, $D_{ii} g > 0$, $i = 2, 3, 4, 5, 6$, in E. Let $p_0, q_0 : [0, \beta) \to \mathbb{R}$ be differentiable functions such that

$$\begin{aligned} p_0''(t) &< g(t, p_0(t), p_0'(t), p_0([t]), p_0'([t]), p_0''([t])), \\ p_0(0) &= \alpha_0, \quad p_0'(0) = \alpha_1, \quad p_0''(0) = \alpha_2 \end{aligned} \tag{14}$$

and

$$q_0''(t) > g(t, q_0(t), q_0'(t), q_0([t]), q_0'([t]), q_0''([t])),\\ q_0(0) = \alpha_0, \quad q_0'(0) = \alpha_1, \quad q_0''(0) = \alpha_2 \tag{15}$$

respectively. Then there exists a Chaplygin sequence $\{(p_n, q_n)\}$ such that

$$p_n(t) < p_{n+1}(t) < y(t) < q_{n+1}(t) < q_n(t), \quad t \in [0, \beta],$$

and

$$p_n(0) = \alpha_0 = q_n(0), \quad p_n'(0) = \alpha_1 = q_n'(0), \quad p_n''(0) = \alpha_2 = q_n''(0),$$

where y is the unique solution of (1)–(2) and the sequences of functions $\{p_n\}$ and $\{q_n\}$ converge uniformly on $[0, \beta]$ to y.

Proof Define

$$\tilde{\beta} = \begin{cases} [\beta] + 1 & \text{if } \beta \neq [\beta], \\ \beta & \text{if } \beta = [\beta]. \end{cases} \tag{16}$$

From the hypothesis, we see that p_0 and q_0 are, respectively, the lower and upper solutions of (1)–(2). By Lemma 3, we have

$$p_0(t) \leq y(t) \leq q_0(t), \quad t \in \bigcup_{r=0}^{\tilde{\beta}-2} [r, r+1) \cup [\tilde{\beta} - 1, \tilde{\beta})$$

with

$$p_0(r) = y(r) = q_0(r), \quad p_0'(r) = y'(r) = q_0'(r), \quad p_0''(r) = y''(r) = q_0''(r),$$

for each $r = 0, 1, \ldots, \tilde{\beta} - 2$. For $t \in \bigcup_{r=0}^{\tilde{\beta}-2} [r, r+1) \cup [\tilde{\beta} - 1, \tilde{\beta})$, we define $g_1 : E \to \mathbb{R}$ and $g_2 : E \to \mathbb{R}$ as follows.

$$\begin{aligned} g_1(t, y(t), y'(t), y([t]), y'([t]), y''([t]); p_0, q_0) &= g(t, p_0(t), p_0'(t), p_0([t]), p_0'([t]), p_0''([t])) \\ &+ \frac{1}{5} D_2 g(t, p_0(t), p_0'(t), p_0([t]), p_0'([t]), p_0''([t]))(y(t) - p_0(t)) \\ &+ \frac{1}{5} D_3 g(t, p_0(t), p_0'(t), p_0([t]), p_0'([t]), p_0''([t]))(y'(t) - p_0'(t)) \\ &+ \frac{1}{5} D_4 g(t, p_0(t), p_0'(t), p_0([t]), p_0'([t]), p_0''([t]))(y([t]) - p_0([t])) \\ &+ \frac{1}{5} D_5 g(t, p_0(t), p_0'(t), p_0([t]), p_0'([t]), p_0''([t]))(y'([t]) - p_0'([t])) \\ &+ \frac{1}{5} D_6 g(t, p_0(t), p_0'(t), p_0([t]), p_0'([t]), p_0''([t]))(y''([t]) - p_0''([t])) \end{aligned} \tag{17}$$

and

$$g_2(t, y(t), y'(t), y([t]), y'([t]), y''([t]); p_0, q_0) = g(t, p_0(t), p_0'(t), p_0([t]), p_0'([t]), p_0''([t]))$$
$$+\frac{1}{5}\{g(t, p_0(t), p_0'(t), p_0([t]), p_0'([t]), p_0''([t])) - g(t, q_0(t), q_0'(t), q_0([t]), q_0'([t]), q_0''([t]))\}$$
$$\left\{\frac{y(t) - p_0(t)}{p_0(t) - q_0(t)} + \frac{y'(t) - p_0'(t)}{p_0'(t) - q_0'(t)} + \frac{y([t]) - p_0([t])}{p_0([t]) - q_0([t])} + \frac{y'([t]) - p_0'([t])}{p_0'([t]) - q_0'([t])} + \frac{y''([t]) - p_0''([t])}{p_0''([t]) - q_0''([t])}\right\}. \tag{18}$$

We find that if $t = r$, where $r = 0, 1, 2, \ldots, \tilde{\beta} - 1$, then

$$\begin{aligned} &g_1(t, y(t), y'(t), y([t]), y'([t]), y''([t]); p_0, q_0) \\ &= g_2(t, y(t), y'(t), y([t]), y'([t]), y''([t]); p_0, q_0). \end{aligned}$$

For $t \in \bigcup_{r=0}^{\tilde{\beta}-2}[r, r+1)$, let p_1 and q_1 be solutions of the linear neutral differential equations

$$\begin{aligned} &y''(t) = g_1(t, y(t), y'(t), y([t]), y'([t]), y''([t]); p_0, q_0), \\ &y(r) = p_0(r), \quad y'(r) = p_0'(r), \quad y''(r) = p_0''(r), \end{aligned} \tag{19}$$

and

$$\begin{aligned} &y''(t) = g_2(t, y(t), y'(t), y([t]), y'([t]), y''([t]); p_0, q_0), \\ &y(r) = q_0(r), \quad y'(r) = q_0'(r), \quad y''(r) = q_0''(r), \end{aligned} \tag{20}$$

respectively. Since p_0 is a lower solution, using Eq. (17), we get

$$\begin{aligned} p_0''(t) &< g(t, p_0(t), p_0'(t), p_0([t]), p_0'([t]), p_0''([t])) \\ &= g_1(t, p_0(t), p_0'(t), p_0([t]), p_0'([t]), p_0''([t]); p_0, q_0). \end{aligned} \tag{21}$$

Now, using (19) and (21), and in view of Lemma 3, we obtain

$$p_0(t) \le p_1(t), \quad t \in (r, r+1). \tag{22}$$

On the same lines, we can show that

$$q_1(t) \le q_0(t), \quad t \in (r, r+1). \tag{23}$$

Also, from Eq. (18)

$$\begin{aligned} p_0''(t) &< g(t, p_0(t), p_0'(t), p_0([t]), p_0'([t]), p_0''([t])) \\ &= g_2(t, p_0(t), p_0'(t), p_0([t]), p_0'([t]), p_0''([t]); p_0, q_0), \quad t \in \bigcup_{r=0}^{\tilde{\beta}-2}[r, r+1) \end{aligned}$$

and, from Eq. (20)

$$q_1''(t) = g_2(t, q_1(t), q_1'(t), q_1([t]), q_1'([t]), q_1''([t])), \quad t \in \bigcup_{r=0}^{\tilde{\beta}-2} [r, r+1).$$

By Lemma 3,

$$p_0(t) \le q_1(t), \quad t \in [r, r+1), \quad r = 0, 1, 2, \ldots, \tilde{\beta} - 2. \tag{24}$$

Now, we show that

$$p_1''(t) < g(t, p_1(t), p_1'(t), p_1([t]), p_1'([t]), p_1''([t])), \quad t \in (r, r+1), \quad r = 0, 1, 2, \ldots, \tilde{\beta} - 2. \tag{25}$$

First we note that for $t \in [r, r+1)$, $r = 0, 1, 2, \ldots, \tilde{\beta} - 2$,

$$p_0(t) \le p_1(t), \quad p_0'(t) \le p_1'(t), \quad p_0([t]) \le p_1([t]), \quad p_0'([t]) \le p_1'([t]), \quad p_0''([t]) \le p_1''([t]).$$

It follows from the Mean Value Theorem that there exist ξ_i, $i = 1, 2, 3, 4, 5$ such that for $t \in [r, r+1)$, $r = 0, 1, 2, \ldots, \tilde{\beta} - 2$, $p_0(t) \le \xi_1 \le p_1(t)$, $p_0'(t) \le \xi_2 \le p_1'(t)$, $p_0([t]) \le \xi_3 \le p_1([t])$, $p_0'([t]) \le \xi_4 \le p_1'([t])$, $p_0''([t]) \le \xi_5 \le p_1''([t])$ and

$$\begin{aligned}
g(t, p_1(t), p_1'(t), p_1([t]), p_1'([t]), p_1''([t])) &= g(t, p_0(t), p_0'(t), p_0([t]), p_0'([t]), p_0''([t])) \\
&+ \frac{1}{5} D_2 g(t, p_0(t), p_0'(t), p_0([t]), p_0'([t]), p_0''([t]))(p_1(t) - p_0(t)) \\
&+ \frac{1}{5} D_3 g(t, p_0(t), p_0'(t), p_0([t]), p_0'([t]), p_0''([t]))(p_1'(t) - p_0'(t)) \\
&+ \frac{1}{5} D_4 g(t, p_0(t), p_0'(t), p_0([t]), p_0'([t]), p_0''([t]))(p_1([t]) - p_0([t])) \\
&+ \frac{1}{5} D_5 g(t, p_0(t), p_0'(t), p_0([t]), p_0'([t]), p_0''([t]))(p_1'([t]) - p_0'([t])) \\
&+ \frac{1}{5} D_6 g(t, p_0(t), p_0'(t), p_0([t]), p_0'([t]), p_0''([t]))(p_1''([t]) - p_0''([t])) \\
&+ \frac{1}{10} D_{22} g(t, \xi_1, p_0'(t), p_0([t]), p_0'([t]), p_0''([t]))(p_1(t) - p_0(t))^2 \\
&+ \frac{1}{10} D_{33} g(t, p_0(t), \xi_2, p_0([t]), p_0'([t]), p_0''([t]))(p_1'(t) - p_0'(t))^2 \\
&+ \frac{1}{10} D_{44} g(t, p_0(t), p_0'(t), \xi_3, p_0'([t]), p_0''([t]))(p_1([t]) - p_0([t]))^2 \\
&+ \frac{1}{10} D_{55} g(t, p_0(t), p_0'(t), p_0([t]), \xi_4, p_0''([t]))(p_1'([t]) - p_0'([t]))^2 \\
&+ \frac{1}{10} D_{66} g(t, p_0(t), p_0'(t), p_0([t]), p_0'([t]), \xi_5)(p_1''([t]) - p_0''([t]))^2.
\end{aligned}$$

Since $D_{ii} g > 0$, $i = 2, 3, 4, 5, 6$, we obtain

$$\begin{aligned} g(t,p_1(t),p_1'(t),p_1([t]),p_1'([t]),p_1''([t])) &> g(t,p_0(t),p_0'(t),p_0([t]),p_0'([t]),p_0''([t])) \\ &+ \frac{1}{5}D_2 g(t,p_0(t),p_0'(t),p_0([t]),p_0'([t]),p_0''([t]))(p_1(t)-p_0(t)) \\ &+ \frac{1}{5}D_3 g(t,p_0(t),p_0'(t),p_0([t]),p_0'([t]),p_0''([t]))(p_1'(t)-p_0'(t)) \\ &+ \frac{1}{5}D_4 g(t,p_0(t),p_0'(t),p_0([t]),p_0'([t]),p_0''([t]))(p_1([t])-p_0([t])) \\ &+ \frac{1}{5}D_5 g(t,p_0(t),p_0'(t),p_0([t]),p_0'([t]),p_0''([t]))(p_1'([t])-p_0'([t])) \\ &+ \frac{1}{5}D_6 g(t,p_0(t),p_0'(t),p_0([t]),p_0'([t]),p_0''([t]))(p_1''([t])-p_0''([t])). \end{aligned}$$

That is,

$$g(t,p_1(t),p_1'(t),p_1([t]),p_1'([t]),p_1''([t])) > g_1(t,p_1(t),p_1'(t),p_1([t]),p_1'([t]),p_1''([t])),$$

for all $t \in [r, r+1)$, $r = 0, 1, 2, \ldots, \tilde{\beta} - 2$.

Since p_1 is a solution of (19), we get that

$$p_1''(t) < g(t,p_1(t),p_1'(t),p_1([t]),p_1'([t]),p_1''([t])), \quad t \in [r,r+1), \quad r = 0,1,2,\ldots,\tilde{\beta}-2.$$

Thus, p_1 is a lower function and hence

$$p_1(t) \leq y(t), \quad t \in [r, r+1), r = 0, 1, 2, \ldots, \tilde{\beta} - 2. \tag{26}$$

Now, since q_1 is a solution of (20), using Eq. (18), we can write

$$q_1''(t) > g(t, q_1(t), q_1'(t), q_1([t]), q_1'([t]), q_1''([t]); p_0, q_0), \quad t \in [r, r+1), r = 0, 1, 2, \ldots, \tilde{\beta} - 2.$$

Thus, q_1 is an upper function and hence

$$y(t) \leq q_1(t), \quad t \in [r, r+1), r = 0, 1, 2, \ldots, \tilde{\beta} - 2. \tag{27}$$

From (26) and (27) and in view of (22) and (23), we get

$$p_0(t) \leq p_1(t) \leq y(t) \leq q_1(t) \leq q_0(t) \quad \text{on} \quad t \in (r, r+1), r = 0, 1, 2, \ldots, \tilde{\beta} - 2. \tag{28}$$

Based on the above discussion, we see that for a given pair of functions (p_0, q_0), we obtain a new pair of functions (p_1, q_1) such that

$$p_1''(t) < g(t, p_1(t), p_1'(t), p_1([t]), p_1'([t]), p_1''([t]))$$

and

$$q_1''(t) > g(t, q_1(t), q_1'(t), q_1([t]), q_1'([t]), q_1''([t]))$$

for $t \in [r, r+1), r = 0, 1, 2, \ldots, \tilde{\beta} - 2$ with $p_1(r) = \alpha_0 = q(r)$, $p_1'(r) = \alpha_1 = q_1'(r)$, $p_1([r]) = \alpha_0 = q_1([r])$, $p_1'([r]) = \alpha_1 = q_1'([r])$, $p_1''([r]) = \alpha_2 = q_1''([r])$. Repeating this process we obtain a well-defined *Chaplygin Sequence* $\{(p_{n+1}, q_{n+1})\}$ of functions such that

(i) $p_n''(t) < g(t, p_n(t), p_n'(t), p_n([t]), p_n'([t]), p_n''([t]))$, for $t \in [r, r+1), r = 0, 1, 2, \ldots, \tilde{\beta} - 2$ and $p_n(r) = \alpha_0$, $p_n'(r) = \alpha_1$, $p_n([r]) = \alpha_0$, $p_n'([r]) = \alpha_1$, $p_n''([r]) = \alpha_2$.

(ii) $q_n''(t) > g(t, q_n(t), q_n'(t), q_n([t]), q_n'([t]), q_n''([t]))$, for $t \in [r, r+1), r = 0, 1, 2, \ldots, \tilde{\beta} - 2$ and $q_n(r) = \alpha_0$, $q_n'(r) = \alpha_1$, $q_n([r]) = \alpha_0$, $q_n'([r]) = \alpha_1$, $q_n''([r]) = \alpha_2$.

(iii) $p_n(t) \le p_{n+1}(t) \le y(t) \le q_{n+1}(t) \le q_n(t), \quad t \in [\tilde{\beta} - 1, \beta]$.

(iv) $p_{n+1}''(t) = g_1(t, p_{n+1}(t),$ $p_{n+1}'(t), p_{n+1}([t]), p_{n+1}'([t]), p_{n+1}''([t]); p_n, q_n), \quad t \in [\tilde{\beta} - 1, \beta]$.

(v) $q_{n+1}''(t) = g_2(t, q_{n+1}(t), q_{n+1}'(t), q_{n+1}([t]), q_{n+1}'([t]),$ $q_{n+1}''([t]); p_n, q_n), \quad t \in [\tilde{\beta} - 1, \beta]$.

From (iii), we see that $\{p_n\}$ and $\{q_n\}$ are monotonic sequences that are uniformly bounded on $[\tilde{\beta} - 1, \beta]$. Furthermore, since each p_n and q_n are solutions of (1)–(2), the sequences $\{p_n\}$ and $\{q_n\}$ are equicontiuous on $[\tilde{\beta} - 1, \beta]$. Finally, application of Theorem 1 yields that the sequences $\{p_n\}$ and $\{q_n\}$ converge uniformly to y on $[\tilde{\beta} - 1, \beta]$. This completes the proof. □

Having established the Chaplygin theorem for second-order neutral differential equations for piecewise constant delay, we shall now establish another fundamental result that actually demonstrates that the results reported in this paper are sharper than those present in existing literature.

Theorem 4 *In addition to the hypothesis of the Theorem 3, define*

$$I := \left\{ (t, y) : p_0(t) \le y \le q_0(t), \quad t \in \bigcup_{r=0}^{\tilde{\beta}-2} [r, r+1) \cup [\tilde{\beta} - 1, \beta) \right\},$$

where $\tilde{\beta}$ is defined in (16). Also, define

$$H := \sup_{(t,y) \in I} \{ D_i g(t, x, y, z, u, v) : i = 2, 3, 4, 5, 6 \}$$

and

$$K := \sup_{(t,y)\in I} \{D_{ii}g(t,x,y,z,u,v) : i = 2,3,4,5,6\}.$$

If $0 \le q_0(t) - p_0(t) \le \dfrac{1}{2K\beta e^{H\beta}} := \varepsilon$, then

$$|q_n(t) - p_n(t)| \le \left(\frac{1}{5}\right)^n \frac{2\epsilon}{2^{2^n}}, \quad t \in [0,\beta], \quad n \in \mathbb{N}_0, \tag{29}$$

where p_n and q_n are, respectively, lower and upper solutions of (1)–(2).

Proof We shall prove the result by the principle of mathematical induction on n. By assumption, we see that the result clearly holds for $n = 0$. Suppose the result holds for a fixed $n \in \mathbb{N}$. That is,

$$|q_n(t) - p_n(t)| \le \left(\frac{1}{5}\right)^n \frac{2\epsilon}{2^{2^n}}.$$

Based on the definition of p_{n+1} and q_{n+1} given in Theorem 3, we can write

$$\begin{aligned}
&|q''_{n+1}(t) - p''_{n+1}(t)| \\
&= |g_2(t, q_{n+1}(t), q'_{n+1}(t), q_{n+1}([t]), q'_{n+1}([t]), q''_{n+1}([t]); p_n, q_n) \\
&\quad - g_1(t, p_{n+1}(t), p'_{n+1}(t), p_{n+1}([t]), p'_{n+1}([t]), p''_{n+1}([t]); p_n, q_n)| \\
&= \Big| g(t, p_n(t), p'_n(t), p_n([t]), p'_n([t]), p''_n([t])) + \frac{1}{5}\{g(t, p_n(t), p'_n(t), p_n([t]), p'_n([t]), p''_n([t])) \\
&\quad - g(t, q_n(t), q'_n(t), q_n([t]), q'_n([t]), q''_n([t]))\}\left\{\frac{q_{n+1}(t) - p_n(t)}{p_n(t) - q_n(t)} + \frac{q'_{n+1}(t) - p'_n(t)}{p'_n(t) - q'_n(t)}\right. \\
&\quad \left. + \frac{q'_{n+1}([t]) - p'_n([t])}{p'_n([t]) - q'_n([t])} + \frac{q''_{n+1}([t]) - p''_n([t])}{p''_n([t]) - q''_n([t])}\right\} - g(t, p_n(t), p'_n(t), p_n([t]), p'_n([t]), p''_n([t])) \\
&\quad + \frac{1}{5}D_2 g(t, p_n(t), p'_n(t), p_n([t]), p'_n([t]), p''_n([t]))(p_{n+1}(t) - p_n(t)) \\
&\quad + \frac{1}{5}D_3 g(t, p_n(t), p'_n(t), p_n([t]), p'_n([t]), p''_n([t]))(p'_{n+1}(t) - p'_n(t)) \\
&\quad + \frac{1}{5}D_4 g(t, p_n(t), p'_n(t), p_n([t]), p'_n([t]), p''_n([t]))(p_{n+1}([t]) - p_n([t])) \\
&\quad + \frac{1}{5}D_5 g(t, p_n(t), p'_n(t), p_n([t]), p'_n([t]), p''_n([t]))(p'_{n+1}([t]) - p'_n([t])) \\
&\quad + \frac{1}{5}D_6 g(t, p_n(t), p'_n(t), p_n([t]), p'_n([t]), p''_n([t]))(p''_{n+1}([t]) - p''_n([t]))\Big|.
\end{aligned}$$

It follows from Mean Value Theorem that there exists $\eta_n(t) \in (p_n(t), q_n(t))$, $n \in \mathbb{N}$ such that $p'_n(t) \le \eta'_n(t) \le q'_n(t)$, $p_n([t]) \le \eta_n([t]) \le q_n([t])$, $p'_n([t]) \le \eta'_n([t]) \le q'_n([t])$, $p''_n([t]) \le \eta''_n([t]) \le q''_n([t])$ and

$$\begin{aligned}
&|q''_{n+1}(t) - p''_{n+1}(t)| \\
&\quad \le \frac{1}{5}\Big| D_2 g(t, \eta_n(t), \eta'_n(t), \eta_n([t]), \eta'_n([t]), \eta''_n([t]))(q_{n+1}(t) - p_n(t)) \\
&\qquad + D_3 g(t, \eta_n(t), \eta'_n(t), \eta_n([t]), \eta'_n([t]), \eta''_n([t]))(q'_{n+1}(t) - p'_n(t)) \\
&\qquad + D_4 g(t, \eta_n(t), \eta'_n(t), \eta_n([t]), \eta'_n([t]), \eta''_n([t]))(q_{n+1}([t]) - p_n([t])) \\
&\qquad + D_5 g(t, \eta_n(t), \eta'_n(t), \eta_n([t]), \eta'_n([t]), \eta''_n([t]))(q'_{n+1}([t]) - p'_n([t])) \\
&\qquad + D_6 g(t, \eta_n(t), \eta'_n(t), \eta_n([t]), \eta'_n([t]), \eta''_n([t]))(q''_{n+1}([t]) - p''_n([t])) \\
&\qquad - D_2 g(t, p_n(t), p'_n(t), p_n([t]), p'_n([t]), p''_n([t]))(p_{n+1}(t) - p_n(t)) \\
&\qquad - D_3 g(t, p_n(t), p'_n(t), p_n([t]), p'_n([t]), p''_n([t]))(p'_{n+1}(t) - p'_n(t)) \\
&\qquad - D_4 g(t, p_n(t), p'_n(t), p_n([t]), p'_n([t]), p''_n([t]))(p_{n+1}([t]) - p_n([t])) \\
&\qquad - D_5 g(t, p_n(t), p'_n(t), p_n([t]), p'_n([t]), p''_n([t]))(p'_{n+1}([t]) - p'_n([t])) \\
&\qquad - D_6 g(t, p_n(t), p'_n(t), p_n([t]), p'_n([t]), p''_n([t]))(p''_{n+1}([t]) - p''_n([t]))\Big|.
\end{aligned}$$

Since $D_{ii}g > 0$, $i = 2, 3, 4, 5, 6$, each $D_i g$ is strictly increasing. Also, since $\eta_n(t) > p_n(t)$, $\eta'_n(t) \ge p'_n(t)$, $\eta_n([t]) \ge p_n([t])$, $\eta'_n([t]) \ge p'_n([t])$, $\eta''_n([t]) \ge p''_n([t])$, it follows that for $i = 2, 3, 4, 5, 6$,

$$D_i g(t, \eta_n(t), \eta'_n(t), \eta_n([t]), \eta'_n([t]), \eta''_n([t])) > D_i g(t, p_n(t), p'_n(t), p_n([t]), p'_n([t]), p''_n([t])).$$

Therefore

$$\begin{aligned}
&|q''_{n+1}(t) - p''_{n+1}(t)| \\
&\le \frac{1}{5}\Big| D_2 g(t, \eta_n(t), \eta'_n(t), \eta_n([t]), \eta'_n([t]), \eta''_n([t]))(q_{n+1}(t) - p_n(t)) \\
&\quad + D_3 g(t, \eta_n(t), \eta'_n(t), \eta_n([t]), \eta'_n([t]), \eta''_n([t]))(q'_{n+1}(t) - p'_n(t)) \\
&\quad + D_4 g(t, \eta_n(t), \eta'_n(t), \eta_n([t]), \eta'_n([t]), \eta''_n([t]))(q_{n+1}([t]) - p_n([t])) \\
&\quad + D_5 g(t, \eta_n(t), \eta'_n(t), \eta_n([t]), \eta'_n([t]), \eta''_n([t]))(q'_{n+1}([t]) - p'_n([t])) \\
&\quad + D_6 g(t, \eta_n(t), \eta'_n(t), \eta_n([t]), \eta'_n([t]), \eta''_n([t]))(q''_{n+1}([t]) - p''_n([t])) \\
&\quad + \Big\{ D_2 g(t, \eta_n(t), \eta'_n(t), \eta_n([t]), \eta'_n([t]), \eta''_n([t])) \\
&\quad - D_2 g(t, p_n(t), p'_n(t), p_n([t]), p'_n([t]), p''_n([t])) \Big\}(p_{n+1}(t) - p_n(t)) \\
&\quad + \Big\{ D_3 g(t, \eta_n(t), \eta'_n(t), \eta_n([t]), \eta'_n([t]), \eta''_n([t])) \\
&\quad - D_3 g(t, p_n(t), p'_n(t), p_n([t]), p'_n([t]), p''_n([t])) \Big\}(p'_{n+1}(t) - p'_n(t)) \\
&\quad + \Big\{ D_4 g(t, \eta_n(t), \eta'_n(t), \eta_n([t]), \eta'_n([t]), \eta''_n([t])) \\
&\quad - D_4 g(t, p_n(t), p'_n(t), p_n([t]), p'_n([t]), p''_n([t])) \Big\}(p_{n+1}([t]) - p_n([t])) \\
&\quad + \Big\{ D_5 g(t, \eta_n(t), \eta'_n(t), \eta_n([t]), \eta'_n([t]), \eta''_n([t])) \\
&\quad - D_5 g(t, p_n(t), p'_n(t), p_n([t]), p'_n([t]), p''_n([t])) \Big\}(p'_{n+1}([t]) - p'_n([t])) \\
&\quad + \Big\{ D_6 g(t, \eta_n(t), \eta'_n(t), \eta_n([t]), \eta'_n([t]), \eta''_n([t])) \\
&\quad - D_6 g(t, p_n(t), p'_n(t), p_n([t]), p'_n([t]), p''_n([t])) \Big\}(p''_{n+1}([t]) - p''_n([t]))\Big|.
\end{aligned}$$

In view of the definition of H and the Mean Value Theorem, there exists

$v_n(t) \in (p_n(t), \eta_n(t))$, $n \in \mathbb{N}$ such that $p'_n(t) < v'_n(t) < \eta'_n(t)$, $p_n([t]) < v_n([t]) < \eta_n([t])$, $p'_n([t]) < v'_n([t]) < \eta'_n([t])$, $p''_n([t]) < v''_n([t]) < \eta''_n([t])$, and

$$
\begin{aligned}
&|q''_{n+1}(t) - p''_{n+1}(t)| \\
&\leq \frac{1}{5} H \Big\{ |q_{n+1}(t) - p_{n+1}(t)| + |q'_{n+1}(t) - p'_{n+1}(t)| + |q_{n+1}([t]) - p_{n+1}([t])| \\
&\quad + |q'_{n+1}([t]) - p'_{n+1}([t])| + |q''_{n+1}([t]) - p''_{n+1}([t])| \Big\} \\
&\quad + \frac{1}{5} \Big| D_{22} g(t, v_n(t), v'_n(t), v_n([t]), v'_n([t]), v''_n([t]))(\eta_n(t) - v_n(t))(p_{n+1}(t) - p_n(t)) \\
&\quad + D_{33} g(t, v_n(t), v'_n(t), v_n([t]), v'_n([t]), v''_n([t]))(\eta'_n(t) - v'_n(t))(p'_{n+1}(t) - p'_n(t)) \\
&\quad + D_{44} g(t, v_n(t), v'_n(t), v_n([t]), v'_n([t]), v''_n([t]))(\eta_n([t]) - v_n([t]))(p_{n+1}([t]) - p_n([t])) \\
&\quad + D_{55} g(t, v_n(t), v'_n(t), v_n([t]), v'_n([t]), v''_n([t]))(\eta'_n([t]) - v'_n([t]))(p'_{n+1}([t]) - p'_n([t])) \\
&\quad + D_{66} g(t, v_n(t), v'_n(t), v_n([t]), v'_n([t]), v''_n([t]))(\eta''_n([t]) - v''_n([t]))(p''_{n+1}([t]) - p''_n([t])) \Big|.
\end{aligned}
$$

Again, in view of how K is defined, we obtain

$$
\begin{aligned}
|q''_{n+1}(t) - p''_{n+1}(t)| \leq & \frac{1}{5} H \Big\{ |q_{n+1}(t) - p_{n+1}(t)| + |q'_{n+1}(t) - p'_{n+1}(t)| + |q_{n+1}([t]) - p_{n+1}([t])|. \\
& + |q'_{n+1}([t]) - p'_{n+1}([t])| + |q''_{n+1}([t]) - p''_{n+1}([t])| \Big\} \\
& + \frac{1}{5} K \Big\{ |q_n(t) - p_n(t)|^2 + |q'_n(t) - p'_n(t)|^2 + |q_n([t]) - p_n([t])|^2 \\
& + |q'_n([t]) - p'_n([t])|^2 + |q''_n([t]) - p''_n([t])|^2 \Big\}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mathcal{D}_{2,+} |q_{n+1}(t) - p_{n+1}(t)| &\leq H |q_{n+1}(t) - p_{n+1}(t)| + K |q_n(t) - p_n(t)|^2 \\
&\leq H |q_{n+1}(t) - p_{n+1}(t)| + K \left(\frac{1}{5}\right)^{2n} \frac{2^2 \varepsilon^2}{2^{2^{n+1}}}.
\end{aligned}
$$

Now, employing Theorem 2, we get

$$
\begin{aligned}
|q_{n+1}(t) - p_{n+1}(t)| &\leq K \left(\frac{1}{5}\right)^{2n} \frac{2^2 \varepsilon^2}{2^{2^{n+1}}} \int_r^t e^{H(t-s)} \mathrm{d}s \\
&\leq \left(\frac{1}{5}\right)^{2n} \frac{2^2 \varepsilon^2}{2^{2^{n+1}}} K \beta e^{H\beta} \\
&\leq \left(\frac{1}{5}\right)^{n+1} \frac{2\varepsilon}{2^{2^{n+1}}}, \quad n \in \mathbb{N}.
\end{aligned}
$$

Therefore, by the principle of mathematical induction, we can write

$$|q_n(t) - p_n(t)| \leq \left(\frac{1}{5}\right)^n \frac{2\epsilon}{2^{2^n}}, \quad n \in \mathbb{N}_0, \quad \text{for } t \in [0, \beta].$$

The sharpness of our results are now established completing the proof. □

The following Corollary gives an error bound for the difference between the exact and the lower (or upper) solution of (1)–(2).

Corollary 1 *The absolute error between the exact and the approximate (lower or upper) solutions of* (1)–(2) *is*

$$|y(t) - p_n(t)| \leq \left(\frac{1}{5}\right)^n \frac{2\epsilon}{2^{2^n}} \quad \text{and} \quad |q_n(t) - y(t)| \leq \left(\frac{1}{5}\right)^n \frac{2\epsilon}{2^{2^n}}, \quad n \in \mathbb{N}_0,$$

where y is the exact solution and p_n, q_n are approximate solutions of second-order neutral differential equation with piecewise constant delay (1) with initial conditions (2).

Proof The proof follows from Theorem 4 and the triangle inequality. □

Remark 1 We see that the estimate given by (29) is five-times sharper as compared to that for first-order ordinary differential equations, and 1.67 times sharper than that for first-order neutral differential equations with piecewise deviating arguments.

Note 1 As seen in Theorem 3 and Theorem 4, following conditions are necessary:

1. The function g must be continuous and possess continuous second order partial derivatives.
2. The first and second order partial derivatives of g must be bounded.
3. The linear neutral differential equation (see (19) and (20)) must be solvable. (A general procedure to determine the solvability of second-order neutral differential equations may be found in [20]).

In the absence of any of these conditions, Chaplygin's theorem will not hold. As such, these may be considered as the limitations of the method.

4 Conclusion and future scope

This paper extends certain results on upper and lower solutions to second-order neutral differential equations with piecewise constant delay. These results are then used to establish Chaplygin's theorem for second-order neutral differential equations. We finally conclude by showing that the results in the present paper are better than any of those available in existing literature, as we get tighter error bounds.

Since the past few decades, growing interest is seen towards the study of fractional differential equations. While the most commonly used fractional derivatives are the Riemann–Liouville, Caputo, and Grünwald–Letnikov, there are some recently introduced fractional operators (new fractional definitions) such as the generalized fractional derivative (known as Abu-Shady–Kaabar fractional derivative). This fractional definition can obtain the same results as Caputo fractional operator in a very simple way without the need for modified numerical techniques. As Chaplygin's method has not yet been extended to any fractional differential equations, it is an open problem that the work presented here can be studied for fractional differential equations, particularly in the sense of Abu-Shady–Kaabar fractional derivative that has been proposed in [1, 21].

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Declarations

Conflict of interest It is declared that authors has no competing interests.

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